Local WKB construction for boundary Witten Laplacians

Dorian Le Peutrec

Local WKB construction for boundary Witten Laplacians

D. Le Peutrec

IRMAR, UMR-CNRS 6625, Université de Rennes 1,

campus de Beaulieu, 35042 Rennes Cedex, France.

email: `dorian.lepeutrec@univ-rennes1.fr`

October 22, 2008

Abstract

WKB p-forms are constructed as approximate solutions to boundary value problems associated with semi-classical Witten Laplacians. Naturally distorted Neumann or Dirichlet boundary conditions are considered.

MSC 2000: 58J37 (58J10 58J32 81Q20)
Keywords: WKB expansion, Boundary value problem, Witten Laplacian.

1 Introduction

1.1 Motivations

In order to compute accurately the small eigenvalues (i.e. of order $\mathcal{O}(e^{-\frac{C}{h}})$ with $C > 0$) of a self adjoint Witten Laplacian acting on 0-forms,

$$\Delta^{(0)}_{f,h} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x) \ ,$$

as the small parameter $h > 0$ goes to 0 (where the function f is assumed to be a Morse function on some bounded domain Ω with or without boundary), we need WKB approximations of the 1-eigenforms associated with the small eigenvalues of $\Delta^{(1)}_{f,h}$, the Witten Laplacian acting on 1-forms.
In the article of B. Helffer, M. Klein and F. Nier [HKN], the authors worked and constructed local WKB approximations of 1-eigenforms in the case of a manifold without boundary.

According to [ChLi], [HeNi1], and [Lep], the p-eigenforms corresponding to the small eigenvalues of $\Delta_{f,h}^{(p)}$, the self adjoint Witten Laplacian acting on p-forms, concentrate around some *generalized* critical points of f with index p which can belong to the boundary when we consider a self adjoint Witten Laplacian with Neumann or Dirichlet type boundary conditions (in the case of a manifold with boundary).
Hence, in the case with boundary, we need local WKB approximations of 1-eigenforms which concentrate near these generalized critical points like it was done in [HeNi1] for Dirichlet type boundary conditions. Nevertheless, the construction done in [HeNi1] relied on some specific trick which cannot be extended to the construction of local WKB forms in the Neumann case. To treat this last case (see [Lep]), a finer treatment of the three geometries involved in the boundary problem (boundary, metric, Morse function) is carried out.
Moreover, this treatment immediately extends to the construction of local WKB p-forms (for the Neumann case) and it can be extended to the construction of local WKB p-forms for the Dirichlet case by "dual computations". However, only the construction of local WKB p-forms is considered here and the comparison with the corresponding p-eigenforms has only be treated in the case $p = 1$ (in [HeNi1] and [Lep]).

1.2 Main notations

Let $\overline{\Omega}$ be a $\mathcal{C}^\infty$ connected compact oriented Riemannian manifold with boundary $\partial\Omega$ and dimension $n \in \mathbb{N}^*$. We will denote by g_0 the given Riemannian metric on $\overline{\Omega}$; Ω and $\partial\Omega$ will denote respectively its interior and its boundary. The cotangent (resp. tangent) bundle on Ω is denoted by $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle by $\Lambda T^*\Omega = \oplus_{p=0}^{n} \Lambda^p T^*\Omega$ (resp. $\Lambda T\Omega = \oplus_{p=0}^{n} \Lambda^p T\Omega$). The fiber bundles $\Lambda T\partial\Omega = \oplus_{p=0}^{n-1} \Lambda^p T\partial\Omega$ and $\Lambda T^*\partial\Omega = \oplus_{p=0}^{n-1} \Lambda^p T^*\partial\Omega$ are defined similarly. The space of $\mathcal{C}^\infty$, $\mathcal{C}_0^\infty$, L^2, H^s , etc. sections in any of these fiber bundles, E, on $O = \Omega$ or $O = \partial\Omega$, will be denoted respectively by $\mathcal{C}^\infty(O; E)$, $\mathcal{C}_0^\infty(O; E)$, $L^2(O; E)$, $H^s(O; E)$, etc.
When no confusion is possible we will simply use the short notations $\Lambda^p\mathcal{C}^\infty$, $\Lambda^p\mathcal{C}_0^\infty$, $\Lambda^p L^2$ and $\Lambda^p H^s$ for $E = \Lambda^p T^*\Omega$ or $E = \Lambda^p T^*\partial\Omega$.
Note that the L^2 spaces are those associated with the unit volume form for the Riemannian structure on Ω or $\partial\Omega$ (Ω and $\partial\Omega$ are oriented).
The notation $\mathcal{C}^\infty(\overline{\Omega}; E)$ is used for the set of $\mathcal{C}^\infty$ sections up to the boundary.

Let d be the exterior differential on $\mathcal{C}_0^\infty(\Omega; \Lambda T^*\Omega)$,

$$d^{(p)} : \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \,,$$

and d^* its formal adjoint with respect to the L^2-scalar product inherited from the Riemannian structure,

$$d^{(p),*} : \mathcal{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega)\,.$$

Remark 1.2.1. *Note that d and d^* are both well defined on $\mathcal{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$.*

Set, for a function $f \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R})$ and $h > 0$, the distorted operators defined on $\mathcal{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$:

$$d_{f,h} = e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \quad \text{and} \quad d^*_{f,h} = e^{f(x)/h}\,(hd^*)\,e^{-f(x)/h}\,.$$

The Witten Laplacian is the differential operator defined on $\mathcal{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$ by:

$$\Delta_{f,h} = d^*_{f,h} d_{f,h} + d_{f,h} d^*_{f,h} = (d_{f,h} + d^*_{f,h})^2\,. \tag{1.2.1}$$

Remark 1.2.2. *The last equality becomes from the property $dd = d^*d^* = 0$ which implies:*

$$d_{f,h} d_{f,h} = d^*_{f,h} d^*_{f,h} = 0. \tag{1.2.2}$$

It means, by restriction to the p-forms in $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$:

$$\Delta^{(p)}_{f,h} = d^{(p),*}_{f,h} d^{(p)}_{f,h} + d^{(p-1)}_{f,h} d^{(p-1),*}_{f,h}\,.$$

Note that (1.2.2) implies that, for all u in $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$,

$$\Delta^{(p+1)}_{f,h} d^{(p)}_{f,h} u = d^{(p)}_{f,h} \Delta^{(p)}_{f,h} u \tag{1.2.3}$$

and

$$\Delta^{(p-1)}_{f,h} d^{(p-1),*}_{f,h} u = d^{(p-1),*}_{f,h} \Delta^{(p)}_{f,h} u\,. \tag{1.2.4}$$

We end up this section by a few relations with exterior and interior products (respectively denoted by $\wedge$ and $\mathbf{i}$), gradients (denoted by ∇) and Lie derivatives (denoted by $\mathcal{L}$) which will be very useful:

$$\begin{aligned}
(df\wedge)^* &= \mathbf{i}_{\nabla f} \quad (\text{in } L^2(\overline{\Omega}; \Lambda^p T^*\Omega))\,, && (1.2.5)\\
d_{f,h} &= hd + df\wedge\,, && (1.2.6)\\
d^*_{f,h} &= hd^* + \mathbf{i}_{\nabla f}\,, && (1.2.7)\\
d \circ \mathbf{i}_X + \mathbf{i}_X \circ d &= \mathcal{L}_X\,, && (1.2.8)\\
\Delta_{f,h} &= h^2(d+d^*)^2 + |\nabla f|^2 + h\left(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}\right)\,, && (1.2.9)
\end{aligned}$$

where X denotes a vector field on Ω or $\overline{\Omega}$.

Remark 1.2.3. *The operators introduced depend on the Riemannian metric g_0 but we omit this dependence for conciseness.*

Definition 1.2.4. *We denote by $\vec{n}_\sigma$ the outgoing normal at $\sigma \in \partial\Omega$ and by $\vec{n}^*_\sigma$ the 1-form dual to $\vec{n}_\sigma$ for the Riemannian scalar product.*

For any $\omega \in \mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$, the form $\mathbf{t}\omega$ is the element of $\mathcal{C}^\infty(\partial\Omega; \Lambda^p T^*\Omega)$ defined by:

$$(\mathbf{t}\omega)_\sigma(X_1, \ldots, X_p) = \omega_\sigma(X_1^T, \ldots, X_p^T) \ , \ \forall \sigma \in \partial\Omega \ ,$$

with the decomposition into the tangential and normal components to $\partial\Omega$ at σ: $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$.
Moreover,

$$(\mathbf{t}\omega)_\sigma = \mathbf{i}_{\vec{n}_\sigma}(\vec{n}^*_\sigma \wedge \omega_\sigma) \ .$$

The projected form $\mathbf{t}\omega$, which depends on the choice of $\vec{n}_\sigma$ (i.e. on g_0), can be compared with the canonical pull-back $j^*\omega$ associated with the embedding $j : \partial\Omega \to \Omega$. Actually the exact relationship is $j^*\omega = j^*(\mathbf{t}\omega)$.
The normal part of ω on $\partial\Omega$ is defined by:

$$\mathbf{n}\omega = \omega|_{\partial\Omega} - \mathbf{t}\omega \quad \in \mathcal{C}^\infty(\partial\Omega; \Lambda^p T^*\Omega).$$

Definition 1.2.5. *We denote by $\frac{\partial f}{\partial n}(\sigma)$ or $\partial_n f(\sigma)$ the normal derivative of f at σ:*

$$\frac{\partial f}{\partial n}(\sigma) = \partial_n f(\sigma) := \langle \nabla f(\sigma) \,|\, \vec{n}_\sigma \rangle \ .$$

Assumption 1.2.6. *The functions $f \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and $f\big|_{\partial\Omega} \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$ are Morse functions. Moreover, the function f has no critical points on $\partial\Omega$.*

According to [ChLi], [HeNi1], and [Lep], the Neumann realization (resp. the Dirichlet realization) of the Witten Laplacian, denoted by $\Delta^N_{f,h}$ (resp. $\Delta^N_{f,h}$), is the self adjoint realization of $\Delta_{f,h}$ whose domain is

$$D(\Delta^N_{f,h}) = \{\omega \in \Lambda H^2(\Omega) \, , \ \mathbf{n}\omega = 0 \, , \ \mathbf{n} d_{f,h}\omega = 0\}$$
$$(\text{resp.} \quad D(\Delta^D_{f,h}) = \left\{\omega \in \Lambda H^2(\Omega) \, , \ \mathbf{t}\omega = 0 \, , \ \mathbf{t} d^*_{f,h}\omega = 0\right\} \quad) \, .$$

Definition 1.2.7. *A point $U \in \overline{\Omega}$ is called a generalized critical point of f with index p in the Neumann case (resp. in the Dirichlet case) if:*

- *either $U \in \Omega$ and U is a critical point of f with index p ,*
- *or $U \in \partial\Omega$ and U is a critical point with index p of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(U) < 0$ (resp. $U \in \partial\Omega$ and U is a critical point with index $p-1$ of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(U) > 0$).*

Remark 1.2.8. *This convention implies, in the Neumann case (resp. in the Dirichlet case), that for a generalized critical point U with index p,*

$$p \in \{0, \ldots, n-1\} \quad (\textit{resp. } p \in \{1, \ldots, n\})\,.$$

Moreover, according to these references extending to the boundary case the analysis by Witten in [Wit], we know that the dimension of the spectral subspace associated with the small eigenvalues (i.e. smaller than h) of $\Delta_{f,h}^{(p),N}$ (resp. $\Delta_{f,h}^{(p),D}$) is $m_p(f)$, the number of generalized critical points of f with index p, and that the corresponding eigenvectors concentrate around these generalized critical points.
The construction of WKB approximations of these eigenvectors already exists in the case of a manifold without boundary (see [Wit][HeSj4][HKN][Hel2]). We want here to obtain similar results around the generalized critical points on the boundary for the Neumann and Dirichlet cases.

1.3 A few preliminary results

In the sequel, we will work with different coordinate systems and we will often refer to the next definition.

Definition 1.3.1. *Let σ be a point on the boundary $\partial\Omega$. A local adapted coordinate system around σ is a local coordinate system $(x^1, \ldots, x^n) = (x', x^n)$ centered at σ satisfying the following properties:*

i) $dx^1, \ldots, dx^n$ is an orthonormal basis of $T_U^*(\overline{\Omega})$ positively oriented.

ii) The boundary $\partial\Omega$ corresponds locally to $x^n = 0$ and the interior Ω to $x^n < 0$.

iii) $\frac{\partial}{\partial x^n}|_{\partial\Omega} = \vec{n}$, the outgoing normal at the boundary. Moreover, $\frac{\partial}{\partial x^n}$ is unitary and normal to $\{x^n = \textit{Constant}\}$.

Such a coordinate system is more specific that the one provided by the collar theorem in [Sch], [Duf], and [DuSp]. Moreover, the analysis done in [Pet] pp. 117-122 leads to the next proposition:

Proposition 1.3.2. *A local coordinate system satisfying Definition 1.3.1 always exists.*

Proof. Consider indeed (see [Pet] pp. 119-120)

$$T\partial\Omega^{\perp} = \left\{v \in T_\sigma\overline{\Omega} \,:\, \sigma \in \partial\Omega,\, v \in (T_\sigma\partial\Omega)^{\perp} \subset T_\sigma\overline{\Omega}\right\}\,,$$

where $(T_\sigma\partial\Omega)^\perp$ is the orthogonal complement of $T\partial\Omega$ in $T_\sigma\overline{\Omega}$ (so for each $\sigma\in\partial\Omega$, $T_\sigma\overline{\Omega}=T_\sigma\partial\Omega\oplus^\perp(T_\sigma\partial\Omega)^\perp$). Then, the map $\exp^\perp$ introduced in [Pet] is a diffeomorphism from an open neighborhood of the zero section in $T\partial\Omega^\perp$ onto its image in $\overline{\Omega}$. It means, choosing a point σ near the boundary $\partial\Omega$, that there exists an unique geodesic ν joining σ to a point σ_b on the boundary which satisfies $\dot\nu(\sigma_b)\in T\partial\Omega^\perp$. It is equivalent to say that there exists an unique geodesic ν joining σ to σ_b with $\dot\nu(\sigma_b)=\vec n_{\sigma_b}$.
Set now $-x^n$ the geodesic distance to $\partial\Omega$ and take x' such that $x'\big|_{\partial\Omega}$ is a coordinate system on the boundary and x' is constant along the geodesics parametrized by x^n. The second point of the definition is then satisfied and $\frac{\partial}{\partial x^n}$ is unitary. Moreover, the choice of $x'\big|_{\partial\Omega}$ is arbitrary and we can choose it centered at U such that $dx^1,\dots,dx^n$ is an orthonormal basis of $T_U^*(\overline{\Omega})$ positively oriented. Then the first point of the definition is also satisfied.
Verify now that the third point of the definition is fulfilled. Write

$$\begin{aligned}
\frac{\partial}{\partial x^n}\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma &= \langle\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma+\langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^i}\rangle_\sigma\\
&= 0+\langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^i}\rangle_\sigma\\
&= \langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^n}\rangle_\sigma\\
&= \frac{1}{2}\frac{\partial}{\partial x^i}\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^n}\rangle_\sigma=0\,,
\end{aligned}$$

where we used the fact that ∇ is the Levi-Civita connexion and $\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n}=0$ since x^n is a geodesic curve. Hence,

$$\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma=\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_{\sigma_b}=\langle\vec n_{\sigma_b}\,|\,\frac{\partial}{\partial x^i}\rangle_{\sigma_b}=0\,,$$

which gives the third point of the definition.

∎

Remark 1.3.3. *In a local adapted coordinate system (x',x^n) around σ, remark that the metric g_0 writes*

$$g_0(x)=d(x^n)^2+\sum_{1\le i,j<n}g_{ij}(x)dx^idx^j\,.$$

Moreover, it can be convenient to work with matrices and we note $G_0(x)=(g_{ij}(x))_{ij}$, $G_0^{-1}(x)=(g^{ij}(x))_{ij}$ (remember that $g_{ij}=\langle\frac{\partial}{\partial x^i}\,|\,\frac{\partial}{\partial x^j}\rangle$,

$g^{ij} = \langle dx^i \,|\, dx^j \rangle$, and $dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}$).
Hence, $G_0^{\pm 1}(x)$ has the form, in the coordinate system (x', x^n):

$$G_0^{\pm 1}(x) = \begin{pmatrix} & & & 0 \\ & G_0^{\pm 1'}(x) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} ,$$

with $G_0^{\pm 1}(0) = I_n$.

Lemma 1.3.4. *1) Let be $f_1 \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and $U \in \partial\Omega$ a critical point of $f_1|_{\partial\Omega}$ with $\frac{\partial f_1}{\partial n}(U) \neq 0$. Assume furthermore $\alpha \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$ be a local solution to $|\nabla_T \alpha|^2 = |\nabla_T f_1|^2$ around U.*
Then there exists a neighborhood $\mathcal{V}$ of U in $\overline{\Omega}$ such that the eikonal equation

$$|\nabla \Phi_\pm|^2 = |\nabla f_1|^2 \tag{1.3.1}$$

(on the boundary, it means $|\nabla \Phi_\pm|^2 = |\partial_n \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2$; see the details in the proof)
with the boundary conditions

$$\Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \alpha \quad , \quad \partial_n \Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \pm\frac{\partial f_1}{\partial n}|_{\partial\Omega\cap\mathcal{V}}$$

admits a unique local smooth real-valued solution.

2) There exist local coordinates $(\overline{x}^1, \ldots, \overline{x}^n) = (\overline{x}', \overline{x}^n)$ in a neighborhood of U in $\overline{\Omega}$ with $(\overline{x}', \overline{x}^n)(U) = 0$ where the function $\Phi_\pm$ and the metric g_0 have the form:

$$\Phi_\pm = \mp \overline{x}^n + \alpha(\overline{x}') \quad \text{and} \quad g_0 = g_{nn}(\overline{x})\; d\,(\overline{x}^n)^2 + \sum_{i,j=1}^{n-1} g_{ij}(\overline{x}) d\overline{x}^i d\overline{x}^j \;.$$

Moreover, the boundary $\partial\Omega$ is locally defined by $\{\overline{x}^n = 0\}$ and Ω corresponds to $\left\{ sgn\left(\frac{\partial f_1}{\partial n}(U)\right) \overline{x}^n > 0 \right\}$.

Proof. 1) Take a local adapted coordinate system (x', x^n) around U in order to write (1.3.1):

$$|\partial_{x^n} \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2 = |\partial_{x^n} f_1|^2 + |\nabla_T f_1|^2$$

(see Appendix A.1 for the exact meaning of ∇_T in the interior). We obtain in particular on the boundary,

$$|\partial_n \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2 = |\partial_n f_1|^2 + |\nabla_T \alpha|^2 \;.$$

The first point is then a direct consequence of the Hamilton-Jacobi theorem, due to the condition $\frac{\partial f_1}{\partial n}(U) \neq 0$.
2) Like in [HeSj4], set:

$$f_+ = \Phi_+ - \Phi_- \qquad \text{and} \qquad f_- = \Phi_+ + \Phi_- \ ,$$

and note the relations:

$$\Phi_- = -\frac{1}{2}f_+ + \frac{1}{2}f_- \ , \quad \Phi_+ = \frac{1}{2}f_+ + \frac{1}{2}f_- \ , \tag{1.3.2}$$

$$\nabla f_+ \cdot \nabla f_- = 0 \ , \tag{1.3.3}$$

$$f_+|_{\partial\Omega\cap\mathcal{V}} = 0 \ , \quad \frac{\partial f_+}{\partial n}|_{\partial\Omega\cap\mathcal{V}} = 2\frac{\partial f_1}{\partial n}|_{\partial\Omega\cap\mathcal{V}} \neq 0 \ , \tag{1.3.4}$$

$$\text{and} \qquad f_-|_{\partial\Omega\cap\mathcal{V}} = 2\alpha \ , \quad \frac{\partial f_-}{\partial n}|_{\partial\Omega\cap\mathcal{V}} = 0 \ . \tag{1.3.5}$$

Let $(\overline{x}^1, \ldots, \overline{x}^{n-1}) = \overline{x}'$ denote a set of coordinates on $\partial\Omega$ in a neighborhood of U (then contained in $\mathcal{V}$) and such that $\overline{x}^j(U) = 0$. We extend them in a neighborhood of U in $\overline{\Omega}$ as constant along the integral curve of the vector field ∇f_+. Then we take $\overline{x}^n = -\frac{1}{2}f_+(x)$ for the last coordinate.
In these coordinates, the functions $\Phi_\pm$ and the metric g_0 have the forms announced in the lemma.
We remark furthermore, by (1.3.4) and $\frac{\partial f_1}{\partial n}(U) \neq 0$, that the boundary $\partial\Omega$ is locally defined by $\{\overline{x}^n = 0\}$ and Ω corresponds to $\left\{sgn\left(\frac{\partial f_1}{\partial n}(U)\right)\overline{x}^n > 0\right\}$. ∎

In the sequel, we will apply the first result of this lemma in the Neumann case (resp. in the Dirichlet case) in order to introduce the Agmon distance (associated with the function f) to a generalized critical point U with index p on the boundary.
Then, using the second result of this lemma and Proposition 3.2.11 of [Lep] (resp. Proposition 3.3.9 of [HeNi1]), $\Delta_{f,h}^{(p),N}$ (resp. $\Delta_{f,h}^{(p),D}$) can be viewed locally in $\mathcal{V}$ around $U \in \partial\Omega$ as $\mathcal{A}_N^{(p)}\big|_{\mathcal{V}}$ (resp. as $\mathcal{A}_D^{(p)}\big|_{\mathcal{V}}$) where $\mathcal{A}_N^{(p)}$ (resp. $\mathcal{A}_D^{(p)}$) is a self adjoint Witten Laplacian on $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$ (ever if it means choosing $-\overline{x}^n$ instead of $\overline{x}^n$) whose domain is

$$\begin{aligned} D(\mathcal{A}_N) &= \left\{\omega \in \Lambda H^2(\mathbb{R}^n_-)\,,\ \mathbf{n}\omega = \mathbf{n}d_{f,h}\omega = 0\right\} \\ (\text{resp.} \quad D(\mathcal{A}_D) &= \left\{\omega \in \Lambda H^2(\mathbb{R}^n_-)\,,\ \mathbf{t}\omega = \mathbf{t}d^*_{f,h}\omega = 0\right\} \quad)\,, \end{aligned}$$

and which satisfies

$$\dim\ \mathrm{Ker}\ \mathcal{A}_N^{(p)} = 1 \ \text{ and } \ \sigma(\mathcal{A}_N^{(p)}) \setminus \{0\} \subset \left[Ch^{6/5}, +\infty\right) \tag{1.3.6}$$

$$(\text{resp.} \dim\ \mathrm{Ker}\ \mathcal{A}_D^{(p)} = 1 \ \text{ and } \ \sigma(\mathcal{A}_D^{(p)}) \setminus \{0\} \subset \left[Ch^{6/5}, +\infty\right)). \tag{1.3.7}$$

2 WKB construction near the boundary for $\Delta^{(p)}_{f,h}$, with p in $\{0,\dots,n\}$

2.1 Local WKB construction in the Neumann case

Let U be a critical point with index $p \in \{0,\dots,n-1\}$ of $f\big|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U) < 0$ and take a local adapted coordinate system (x', x^n) around U.

Let φ be the Agmon distance to U on the boundary (i.e. associated with the metric $|\nabla_{x'} f(x',0)|^2\ dx'^2$). We recall that, on the boundary,

$$|\nabla_T f|^2 = |\nabla \varphi|^2$$

and that φ is smooth near U (see [HeSj1]).
We now use the first result of Lemma 1.3.4 with $f_1 = f$ and $\alpha = \varphi$ and we denote by Φ the function Φ_+ of the lemma (Φ is consequently the Agmon distance to U i.e. associated with the metric $|\nabla f(x)|^2\ dx^2$). Hence we have locally:

$$\begin{aligned} |\partial_n \Phi|^2 + |\nabla_T \Phi|^2 &= |\nabla \Phi|^2 = |\nabla f|^2\ , &\quad (2.1.1)\\ \Phi|_{\partial\Omega} &= \varphi\ , &\quad (2.1.2)\\ \partial_n \Phi|_{\partial\Omega} &= \frac{\partial f}{\partial n}|_{\partial\Omega}\ . &\quad (2.1.3) \end{aligned}$$

Moreover, the next relation is valid:

$$\partial^2_{x^n x^n}(f-\Phi)(0) = \partial^2_{nn}(f-\Phi)(0) = 0\ . \qquad (2.1.4)$$

Write indeed in the coordinates (x', x^n), for the metric g_0:

$$|\partial_{x^n}\Phi|^2 + |\nabla_T \Phi|^2_{g_0} = |\partial_{x^n} f|^2 + |\nabla_T f|^2_{g_0}$$

where $|\nabla_T \Phi|^2_{g_0} = \mathcal{O}(|x|^2)$ and $|\nabla_T f|^2_{g_0} = \mathcal{O}(|x|^2)$ because 0 is a critical point of $f\big|_{\partial\Omega}$ in the coordinates (x', x^n) (see indeed for example Appendix A.1).
Apply then ∂_{x^n} to the last equation:

$$\partial_{x^n}\, |\partial_{x^n}\Phi|^2 + \mathcal{O}(|x|) = \partial_{x^n}\, |\partial_{x^n} f|^2 + \mathcal{O}(|x|)$$

i.e., using (2.1.3),

$$2\partial^2_{x^n x^n}(f-\Phi)\partial_{x^n} f = \mathcal{O}(|x|)$$

which yields the result.
According to [HeSj4] pp. 279-280, there exist local coordinates $(\overline{x}', \overline{x}^n)$ centered at U, where $\overline{x}' = (\overline{x}^1, \dots, \overline{x}^{n-1})$ are Morse coordinates for $f\big|_{\partial\Omega}$ around

U, such that $d\overline{x}^1, \dots, d\overline{x}^{n-1}, dx^n$ is orthonormal at U, and

$$f(\overline{x}', 0) = \frac{\lambda_1}{2}(\overline{x}^1)^2 + \cdots + \frac{\lambda_{n-1}}{2}(\overline{x}^{n-1})^2 + f(U) \tag{2.1.5}$$

$$\text{and} \qquad \varphi(\overline{x}') = \frac{|\lambda_1|}{2}(\overline{x}^1)^2 + \cdots + \frac{|\lambda_{n-1}|}{2}(\overline{x}^{n-1})^2 \,. \tag{2.1.6}$$

with $\lambda_i < 0$ for $i \in \{1, \dots, p\}$ and $\lambda_i > 0$ for $i \in \{p+1, \dots, n-1\}$.
Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}$ coincide at U, and even such that $x'\big|_{\partial\Omega} = \overline{x}'\big|_{\partial\Omega}$ since $x'\big|_{\partial\Omega}$ can be chosen freely.

Theorem 2.1.1. *Consider around U a local adapted coordinate system $x = (x', x^n)$ such that $dx^i = d\overline{x}^i$ at U (for i in $\{1, \dots, n-1\}$). There exists locally, in a neighborhood of $x = 0$, a $\mathcal{C}^\infty$ solution u_p^{wkb} to*

$$\begin{aligned} \Delta_{f,h}^{(p)} u_p^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (2.1.7)\\ \mathbf{n} u_p^{wkb} &= 0 \text{ on } \partial\Omega && (2.1.8)\\ \mathbf{n} d_{f,h} u_p^{wkb} &= 0 \text{ on } \partial\Omega \,, && (2.1.9) \end{aligned}$$

where u_p^{wkb} has the form:

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}} \,, \\ \text{with} \quad a(x,h) &\sim \sum_k a^k(x) h^k \ \text{ and } a^0(0) = dx^1 \wedge \cdots \wedge dx^p \,. \end{aligned}$$

2.2 First boundary conditions in the Neumann case

Let us first write, in our coordinate system,

$$a(x,h) = a_I(x,h)\ dx^I = a_{I'}(x,h) dx^{I'} + a_{I_n}(x,h) dx^{I_n} \,, \tag{2.2.1}$$

where $I \in \mathcal{I} := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p \,,\ i_1 < \cdots < i_p\}$,
$I' \in \mathcal{I}' := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p \,,\ i_1 < \cdots < i_p < n\}$,
$I_n \in \mathcal{I}_n := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p \,,\ i_1 < \cdots < i_p = n\}$,
and $dx^{(i_1, \dots, i_p)} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Remark that the Einstein summation convention where repeated indices implies addition has been employed in formula (2.2.1) and, in the sequel, we shall adhere to this notation.
The first boundary condition says only that:

$$\forall I_n \in \mathcal{I}_n \,,\ a_{I_n}((x', 0), h) \sim \sum_k a_{I_n}^k (x', 0) h^k \equiv 0 \tag{2.2.2}$$

which is equivalent to

$$\forall k \in \mathbb{N},\, \forall I_n \in \mathcal{I}_n\,,\ a^k_{I_n}(x',0) \equiv 0\,. \tag{2.2.3}$$

This paragraph specifies some consequences of these conditions.

Proposition 2.2.1. *Using the notations of Appendices A.1 and A.2, the next relations are satisfied for k in $\mathbb{N}$, when (2.2.3) is fulfilled:*

$$\left\{\begin{array}{rcl} \mathbf{t}\left((2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1)a^k\right) & = & (2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id + \mathcal{R}^T_{Neu})a^k_{I'}dx^{I'} + 2\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}da^k \\ \mathbf{n}\left((2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1)a^k\right) & = & 2\left(\frac{\partial a^k_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}+\ell_{I_n}(x',0)\right)dx^{I_n}\,, \end{array}\right.$$

where the ℓ_{I_n}'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) and $\mathcal{R}^T_{Neu}$ is a 0-th order differential operator given on the boundary by the next matrix, in the coordinates (x',x^n):

$$\mathcal{R}^T_{Neu}(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}^{T'}_{Neu}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,Id\,,$$

where $\beta(0)=0$,

$$\gamma(0) = Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,,$$

and

$$\mathcal{R}^{T'}_{Neu}(0) = 2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$

Proposition 2.2.2. *Assume* (2.2.3) *for k in $\mathbb{N}$. The p-form*

$$\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}da^k$$

is then tangential and the next equivalence is locally valid on the boundary $\partial\Omega$:

$$\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}da^k = 0 \Leftrightarrow \mathbf{n}da^k = 0\,.$$

Proof. Write indeed on the boundary $\partial\Omega$:

$$\begin{array}{rcl} \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}da^k & = & \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{n}da^k + \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{t}da^k \\ & = & 0 + \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{n}da^k \\ & = & \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}(da^k)_{I_n}dx^{I_n} \\ & = & (-1)^p\frac{\partial\Phi}{\partial x^n}(da^k)_{I_n}dx^{I_n\setminus\{n\}}\,. \end{array}$$

Moreover, using

$$\frac{\partial f}{\partial n}(U) = \frac{\partial \Phi}{\partial n}(U) < 0\,,$$

$\frac{\partial \Phi}{\partial x^n}$ is locally negative on the boundary which leads to the result.

∎

Lemma 2.2.3. *Under* (2.2.3)*, the next relations are satisfied for k in $\mathbb{N}$:*

$$\left\{\begin{array}{lcl} \mathbf{t}\left((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})a^k\right) & = & \mathbf{t}\left((\mathcal{L}_{\nabla_T\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})a^k\right) = \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}} da^k\,, \\ \mathbf{n}\left((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})a^k\right) & = & \left(\frac{\partial a^k_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \tilde{\ell}_{I_n}(x',0)\right) dx^{I_n}\,, \end{array}\right.$$

where the $\tilde{\ell}_{I_n}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$).

Proof. On the boundary $\partial\Omega$, write the next decomposition:

$$(\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})\,a^k = \mathcal{L}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k + (\mathcal{L}_{\nabla_T\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})\,a^k\,. \tag{2.2.4}$$

Owing to the Cartan formula (1.2.8), rewrite (2.2.4):

$$\begin{aligned} (\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})\,a^k &= \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} da^k + d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k) \\ &+ \mathbf{i}_{(\nabla_T\Phi - \nabla\tilde{\Phi})} da^k + d(\mathbf{i}_{(\nabla_T\Phi - \nabla\tilde{\Phi})} a^k)\,. \end{aligned} \tag{2.2.5}$$

Using Proposition 2.2.2, the first term

$$\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} da^k = \frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}} da^k$$

of the r.h.s. of (2.2.5) is tangential.
Moreover, since $\nabla_T\Phi = \nabla\tilde{\Phi}$ on the boundary (see Appendix A.1), the term $\mathbf{i}_{(\nabla_T\Phi - \nabla\tilde{\Phi})} da^k$ of the r.h.s. equals 0 on $\partial\Omega$.
Hence, write on $\partial\Omega$:

$$(\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})\,a^k = \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} da^k + d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k) + d(\mathbf{i}_{(\nabla_T\Phi - \nabla\tilde{\Phi})} a^k)\,. \tag{2.2.6}$$

Study in a first time the term $d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k)$. Writing

$$a^k = a^k_I dx^I = a^k_{I'} dx^{I'} + a^k_{I_n} dx^{I_n}\,,$$

we deduce (in $\overline{\Omega}$):

$$\begin{aligned} \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k &= a^k_{I_n}\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} dx^{I_n} \\ &= (-1)^{p-1} a^k_{I_n}\frac{\partial\Phi}{\partial x^n} dx^{I_n\setminus\{n\}}\,, \end{aligned}$$

and, applying d to this last relation, we obtain on $\partial\Omega$ (remember that $a^k_{I_n} = 0$ on $\partial\Omega$)

$$\begin{aligned} d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} a^k) &= (-1)^{p-1}\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(a^k_{I_n}\frac{\partial\Phi}{\partial x^n})dx^i\wedge dx^{I_n\setminus\{n\}} \\ &= (-1)^{p-1}\frac{\partial a^k_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}dx^n\wedge dx^{I_n\setminus\{n\}} + 0 \\ &= \frac{\partial a^k_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}dx^{I_n}\,. \end{aligned} \tag{2.2.7}$$

Look now at the third term of the r.h.s. of (2.2.6) and write (remember that $\mathcal{I} \ni I = (i_1, \dots, i_p)$ with $1 \le i_1 \le \cdots \le i_p \le n$ and denote by $\operatorname{ind}(i_k)$ the integer k):

$$\begin{aligned} \mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}a^k_I dx^I &= a^k_I dx^I(\nabla_T\Phi-\nabla\tilde{\Phi}) \\ &= a^k_I\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\left(\nabla_T\Phi-\nabla\tilde{\Phi}\right)_j dx^{I\setminus\{j\}} \\ &= a^k_I\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\alpha_j dx^{I\setminus\{j\}}\,, \end{aligned}$$

where, due to (A.1.2)(A.1.3), for all j in $\{1, \dots, n\}$,

$$\alpha_j = \left(\nabla_T\Phi-\nabla\tilde{\Phi}\right)_j = \sum_{i=1}^{n} g^{ij}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)\,.$$

Moreover, due to the block diagonal form of G_0^{-1}, for all j in $\{1, \dots, n\}$, α_j satisfies (again by (A.1.2)(A.1.3)):

$$\alpha_n(x) \equiv 0 \text{ and } \forall j \in \{1, \dots, n-1\}\,, \alpha_j(x',0) \equiv 0\,.$$

Hence, we obtain on $\partial\Omega$,

$$\begin{aligned} d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}a^k_I dx^I)(x',0) &= \sum_{l=1}^{n}\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^l}(a^k_I\alpha_j)(x',0)dx^l\wedge dx^{I\setminus\{j\}} \\ &= 0 + \sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(a^k_I\alpha_j)(x',0)dx^n\wedge dx^{I\setminus\{j\}} \\ &= \sum_{j\in I'}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(a^k_{I'}\alpha_j)(x',0)dx^n\wedge dx^{I'\setminus\{j\}} \\ &\quad + \sum_{j\in I_n\setminus\{n\}}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(a^k_{I_n}\alpha_j)(x',0)dx^n\wedge dx^{I_n\setminus\{j\}} \\ &= \sum_{j\in I'}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(a^k_{I'}\alpha_j)(x',0)dx^n\wedge dx^{I'\setminus\{j\}}\,, \end{aligned}$$

where we used $\alpha_j(x',0)\equiv 0$ at the second line and $\alpha_n(x)\equiv 0$ at the second to last line.
Using again $\alpha_j(x',0)\equiv 0$ allows us to write on $\partial\Omega$:

$$\begin{aligned} d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}a_I^k dx^I)(x',0) &= a_{I'}^k\sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial\alpha_j}{\partial x^n}(x',0)dx^n\wedge dx^{I'\setminus\{j\}} \\ &= a_{I'}^k\sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+p}\frac{\partial\alpha_j}{\partial x^n}(x',0)dx^{I'\setminus\{j\}}\wedge dx^n \\ &=:\tilde{\ell}_{I_n}(x',0)dx^{I_n}\,, \end{aligned} \tag{2.2.8}$$

where the $\tilde{\ell}_{I_n}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I'}^k$'s (for I' in $\mathcal{I}'$) which do not depend on the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$).
Combining (2.2.6), (2.2.7), and (2.2.8) leads to the result announced in Lemma 2.2.3.

∎

Proof of Proposition 2.2.1.
Remember first the next relation (see indeed Subsection A.2.2):

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*=2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1\,,$$

where $\mathcal{R}_1$ is a 0-th order differential operator. Writing $\mathcal{R}_1=\mathcal{R}_1^T+\mathcal{R}_1^N$, we deduce from Remark A.2.1 (since $a_I^k dx^I=a_{I'}^k dx^{I'}$ on the boundary),

$$\left\{\begin{array}{rcl} \mathbf{t}\left(\mathcal{R}_1(a_I^k dx^I)\right) &=& a_{I'}^k(x',0)\mathcal{R}_1^T(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_1(a_I^k dx^I)\right) &=& a_{I'}^k(x',0)\mathcal{R}_1^N(dx^{I'})=\tilde{\ell}'_{I_n}(x',0)dx^{I_n}\,, \end{array}\right.$$

where the $\tilde{\ell}'_{I_n}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I'}^k$'s (for I' in $\mathcal{I}'$) which do not depend on the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$).

Moreover, $f-\Phi$ satisfies the assumptions of Corollary A.2.5 (from (2.1.1)-(2.1.4)), then $\mathcal{R}_1^T$ is given on the boundary, in the coordinates (x',x^n), by:

$$\mathcal{R}_1^T(x',0)=\begin{pmatrix} & & 0 \\ & \mathcal{R}_1^{T'}(x') & \vdots \\ & & 0 \\ 0 & \cdots \quad 0 & \beta(x') \end{pmatrix}^{(p)}-\gamma(x')\,Id\,,$$

where β, γ are $\mathcal{C}^\infty$ functions which satisfy $\beta(0)=0$,
$\gamma(0)=Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)$, and

$$\mathcal{R}_1^{T'}(0)=2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,.$$

Having in mind Lemma 2.2.3, look now at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1$.
From Proposition A.2.3, write:

$$2\mathcal{L}_{\nabla\tilde{\Phi}} = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_3$$

where $\mathcal{R}_3 = \mathcal{R}_3^T + \mathcal{R}_3^N$ is a 0-th order differential operator such that (since $\tilde{\Phi}$ satisfies the assumptions of Corollary A.2.5),

$$\begin{cases} \mathbf{t}\left(\mathcal{R}_3(a_I^k dx^I)\right) &= a_{I'}^k(x',0)\mathcal{R}_3^T(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_3(a_I^k dx^I)\right) &= a_{I'}^k(x',0)\mathcal{R}_3^N(dx^{I'}) = \tilde{\ell}''_{I_n}(x',0)dx^{I_n}\,, \end{cases}$$

(where the $\tilde{\ell}''_{I_n}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I'}^k$'s (for I' in $\mathcal{I}'$) which do not depend on the $a_{I_n}^k$'s)
and $\mathcal{R}_3^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_3^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_3^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}^{(p)},$$

with

$$\mathcal{R}_3^{T'}(0) = 2\left(\mathrm{Hess}\,(\tilde{\Phi}\big|_{\partial\Omega})(0)\right) = 2\left(\mathrm{Hess}\,(\varphi)(0)\right)$$

(note, according to Remark A.2.4, that the term of index (n,n) of the matrix is indeed 0 since $\frac{\partial^2\tilde{\Phi}}{\partial(x^n)^2} \equiv 0$).

Set $\mathcal{R}_{\mathrm{Neu}} = \mathcal{R}_1 + \mathcal{R}_3$ and $\tilde{\ell}^{(3)}_{I_n} = \tilde{\ell}'_{I_n} + \tilde{\ell}''_{I_n}$ for I_n in $\mathcal{I}_n$. $\mathcal{R}_{\mathrm{Neu}}$ is a 0-th order differential operator which satisfies

$$2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_{\mathrm{Neu}}\,, \tag{2.2.9}$$

and

$$\begin{cases} \mathbf{t}\left(\mathcal{R}_{\mathrm{Neu}}(a_I^k dx^I)\right) &= a_{I'}^k(x',0)\mathcal{R}_{\mathrm{Neu}}^T(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_{\mathrm{Neu}}(a_I^k dx^I)\right) &= a_{I'}^k(x',0)\mathcal{R}_{\mathrm{Neu}}^N(dx^{I'}) = \tilde{\ell}^{(3)}_{I_n}(x',0)dx^{I_n}\,, \end{cases} \tag{2.2.10}$$

where the $\tilde{\ell}^{(3)}_{I_n}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I'}^k$'s (for I' in $\mathcal{I}'$) which do not depend on the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$).
Moreover, $\mathcal{R}_{\mathrm{Neu}}^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_{\mathrm{Neu}}^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_1^{T'}(x',0) + \mathcal{R}_3^{T'}(x',0) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,Id\,,$$

where $\beta(0) = 0$,
$$\gamma(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} - \varphi)(0)\right)\,,$$
and
$$\mathcal{R}_1^{T'}(0) + \mathcal{R}_3^{T'}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$
Look now at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id$. By the Cartan formula (1.2.8),
$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)a^k \;=\; da^k_{I'}(\nabla\tilde{\Phi})dx^{I'} + da^k_{I_n}(\nabla\tilde{\Phi})dx^{I_n}\,,$$
and, using the boundary condition satisfied by the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain:
$$\begin{aligned}(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)a^k &= \sum_{i=1}^{n-1} \frac{\partial a^k_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I'} \\ &= (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)a^k_{I'}dx^{I'}\,. \end{aligned} \tag{2.2.11}$$
Set $\ell_{I_n} = \tilde{\ell}_{I_n} + \frac{1}{2}\tilde{\ell}^{(3)}_{I_n}$ for I_n in $\mathcal{I}_n$. Writing
$$\left(2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1\right)a^k = 2\left(\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}}\right)a^k + \left(2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1\right)a^k\,,$$
and using (2.2.9), (2.2.10), and (2.2.11), we obtain Proposition 2.2.1 after the application of Lemma 2.2.3.

∎

2.3 Proof of Theorem 2.1.1

We shall first consider a WKB-approximation for
$$(\Delta^{(p)}_{f,h} - E(h))u^{wkb}_p = e^{-\frac{\Phi}{h}}\mathcal{O}(h^\infty) \tag{2.3.1}$$
with $E(h) = O(h^2)$ and the boundary conditions (2.1.8)(2.1.9) and then check $E(h) = O(h^\infty)$.
Writing
$$\forall k \in \mathbb{N}\,,\quad d_{f,h}(e^{-\frac{\Phi}{h}}a^k) = e^{-\frac{\Phi}{h}}\left[hda^k + d(f-\Phi)\wedge a^k\right]\,,$$
where a^k and $d(f-\Phi)$ are tangential forms (due to (2.1.8) and (2.1.3)), the second boundary condition corresponds to
$$\mathbf{n}(da^k) = 0 \tag{2.3.2}$$

(and more precisely it says:

$$\forall k \in \mathbb{N}\,, \quad a^k(x) = a^k_{I'}(x)\; dx^{I'} + a^k_{I_n}(x)\; dx^{I_n}$$
$$\text{satisfies} \quad : \quad \forall I' \in \mathcal{I}'\,,\ \frac{\partial a^k_{I'}}{\partial x^n}(x',0) \equiv 0\,).$$

Let us now recall the following relation which will be very useful (see [HeSj4] for a complete proof:

$$e^{\frac{\Phi}{h}}\Delta_{f,h}e^{-\frac{\Phi}{h}} = h^2(d+d^*)^2 + h(\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})\,, \tag{2.3.3}$$

and write, with the notations of Appendix A.2.2,

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = 2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1 = 2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}\,,$$

where $\mathcal{R}$ and $\mathcal{R}_1$ are 0-th differential operators defined in Appendix A.2.2. By looking for $E(h) \sim \sum_{k=1}^{\infty} h^{k+1}E_k$, the interior equation (2.3.1) reads

$$e^{\frac{\Phi}{h}}(\Delta_{f,h} - E(h))e^{-\frac{\Phi}{h}} = h^2[(d+d^*)^2 - h^{-2}E(h)] + h\,[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}]$$

We now verify that it is possible to construct a solution u_p^{wkb} to (2.3.1) in Ω which can be extended to $\overline{\Omega}$ and satisfying the boundary conditions (2.1.8) and (2.1.9).

The construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k's are known on $\partial\Omega$ (see [DiSj],[Hel2]). Actually the non characteristic Cauchy problems

$$[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}]a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega}\,. \tag{2.3.4}$$

are solved by induction with the convention $a_{-1} = 0$.

Hence the problem is reduced to the solving of the system made of the boundary conditions (2.2.3), (2.3.2) and of the compatibility equation on the boundary (see Appendix A.2.2 for the meaning of the notations):

$$[2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1]\,a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{on } \partial\Omega\,. \tag{2.3.5}$$

Owing to Propositions 2.2.1 and 2.2.2 (with the notations of Section 2.2) and to (2.1.3), the system (2.3.5), (2.2.3), (2.3.2) is equivalent to the differential system on $\partial\Omega$:

$$
\begin{cases}
-\mathbf{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{\text{Neu}})a^k_{I'}dx^{I'} & (2.3.6)\\
-\mathbf{n}(d+d^*)^2a^{k-1}-2\ell_{I_n}(x',0)dx^{I_n}=2\dfrac{\partial f}{\partial n}\dfrac{\partial a^k_{I_n}}{\partial x^n}dx^{I_n} & (2.3.7)\\
(2.2.3)\ +\ (2.3.2)\,, &
\end{cases}
$$

where the ℓ_{I_n}'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$).
Moreover, since $dx^i=d\overline{x}^i$ (for $i\in\{1,\dots,n-1\}$) at the point U, thanks to Corollary A.2.5, (2.1.5)-(2.1.6), and according to [HeSj4] pp. 271-274, $\mathcal{R}^T_{\text{Neu}}(0)$ restricted to tangential forms is symmetric with the one dimensional kernel $\mathbb{R}dx^1\wedge\cdots\wedge dx^p$.

Since $a^k_{I'}dx^{I'}$ is tangential and $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id$ only differentiates tangentially the $a^k_{I'}$'s

$$
\left(\quad \text{since } (2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)a^k_{I'}dx^{I'}=\sum_{i=1}^{n-1}\frac{\partial a^k_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I'}\ \right),
$$

(2.3.6) can be rewritten as a tangential system which can be solved according to the analysis of the boundaryless case done in [HeSj4].
Here are the details:
Owing to Proposition 2.2.2, the complete system becomes equivalent to

$$
\begin{cases}
(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{\text{Neu}})a^k_{I'}dx^{I'}=-\mathbf{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}+E_ka^0 & \text{on } \partial\Omega\\
(2\mathcal{L}_{\nabla\Phi}\otimes Id+\mathcal{R})a^k=-(d+d^*)^2a^{k-1} & \text{on } \overline{\Omega}\\
\qquad\forall I_n\in\mathcal{I}_n\,,\ a_{I_n}|_{\partial\Omega}\equiv 0\,. &
\end{cases}
$$

Note that the first line is a degenerate matricial transport equation which can be solved according to [HeSj4][Hel2]:
For $k=0$, take $a^0(0)=dx^1\wedge\cdots\wedge dx^p\in$ Ker $(\mathcal{R}^T_{\text{Neu}}(0))$ and for $k>0$ choose E_k so that the compatibility condition

$$
-\mathbf{t}(d+d^*)^2a^{k-1}(0)+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}(0)+E_ka^0(0)\in\left(\text{ Ker }(\mathcal{R}^T_{\text{Neu}}(0))\right)^\perp
$$

is satisfied. Thus, at every step $k\in\mathbb{N}$, the first and the third line of the previous system fully determine the Cauchy data $a^k(x',0)$ and the number E_k. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Proposition 2.2.2, the second trace condition (2.3.2).

Let us check now $E(h) = \mathcal{O}(h^\infty)$. We prove this by comparing with the half-space problem, for which we know by (1.3.6) that the first eigenvalue is 0 with multiplicity one and that the second one is larger than $Ch^{6/5}$. Take a cut-off function $\chi \in \mathcal{C}_0^\infty(\overline{\Omega})$, $\chi = 1$ in a neighborhood of U such that $\frac{\partial \chi}{\partial n}\big|_{\partial\Omega} = 0$ and set

$$u_p^K = \chi e^{-\frac{\Phi}{h}} \sum_{k=0}^{K} a^k h^k = \chi e^{-\frac{\Phi}{h}} A_h^K \,.$$

From $\frac{\partial \chi}{\partial n}\big|_{\partial\Omega} \equiv 0$ and

$$d_{f,h}(\chi A_h^K) = (hd + df\wedge)\chi A_h^K = hd\chi \wedge A_h^K + \chi d_{f,h} A_h^K \,,$$

the form $u_p^K \in \Lambda^1 H^2(\mathbb{R}^n_-)$ belongs to the domain of $\mathcal{A}_N^{(p)}$ and the approximations u_p^K and $E^K(h) = \sum_{k=1}^K E_k h^{k+1}$ satisfy

$$\left\{\begin{array}{ll} [\mathcal{A}_N^{(p)} - E^K(h)]u_p^K = h^{K+2}\rho^K e^{-\frac{\Phi}{h}} - h^2\,[\Delta,\chi]\,u_p^K = \mathcal{O}(h^{K+2}) & \text{in } \overline{\mathbb{R}^n_-} \\ \mathbf{n}u_p^K = 0 & \text{on } \mathbb{R}^{n-1}\times\{0\} \\ \mathbf{n}d_{f,h}u_p^K = 0 & \text{on } \mathbb{R}^{n-1}\times\{0\}\,, \end{array}\right.$$

for some $\mathcal{C}^\infty$ 1-form ρ^K defined in a neighborhood of U and independent of h.
From $\left\|u_p^{wkb}\right\| \sim ch^{\frac{n+1}{4}}$ (from a direct Laplace method),

$$\left\|u_p^K\right\| \sim ch^{\frac{n+1}{4}}$$

and the spectral theorem then implies that there exists an eigenvalue $\lambda(h)$ of $\mathcal{A}_N^{(p)}$ such that:

$$|E^K(h) - \lambda(h)| = \mathcal{O}(h^{K+2-\frac{n+1}{4}})\,.$$

Choosing the integer number K large enough, the inclusion

$$\sigma(\mathcal{A}_N^{(p)}) \setminus \{0\} \subset [Ch^{6/5}, +\infty)$$

combined with the estimate $E^K(h) = \mathcal{O}(h^2)$ implies $\lambda(h) = 0$. The number K being arbitrary, the construction of the previous quasimode is then possible only if

$$\forall k \in \mathbb{N}^*\,, \quad E_k = 0\,.$$

∎

2.4 Local WKB construction in the Dirichlet case

Let here U be a critical point with index $p-1 \in \{0, \dots, n-1\}$ (i.e. $p \in \{1, \dots, n\}$) of $f|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U) > 0$ and take again a local adapted coordinate system (x', x^n) around U like in Section 2.1.
Let φ be the Agmon distance to U on the boundary and use the first result of Lemma 1.3.4 with $f_1 = f$ and $\alpha = \varphi$. Denoting by Φ the function Φ_- of the lemma (Φ is the Agmon distance to U i.e. associated with the metric $|\nabla f(x)|^2 \, dx^2$), we have locally:

$$\begin{aligned} |\partial_n \Phi|^2 + |\nabla_T \Phi|^2 &= |\nabla \Phi|^2 = |\nabla f|^2 \,, && (2.4.1)\\ \Phi|_{\partial\Omega} &= \varphi \,, && (2.4.2)\\ \partial_n \Phi|_{\partial\Omega} &= -\frac{\partial f}{\partial n}|_{\partial\Omega} \,. && (2.4.3) \end{aligned}$$

Moreover, the next relation is satisfied (see indeed the proof of (2.1.4) and replace $\partial_n \Phi|_{\partial\Omega} = \partial_n f|_{\partial\Omega}$ by $\partial_n \Phi|_{\partial\Omega} = -\partial_n f|_{\partial\Omega}$):

$$\partial^2_{x^n x^n}(f + \Phi)(0) = \partial^2_{nn}(f + \Phi)(0) = 0 \,. \qquad (2.4.4)$$

Like in Section 2.1, there exist other local coordinates $(\overline{x}', \overline{x}^n)$ centered at U, with $\overline{x}' = (\overline{x}^1, \dots, \overline{x}^{n-1})$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}, dx^n$ is orthonormal at U, such that (2.1.5) and (2.1.6) are satisfied with $\lambda_i < 0$ for $i \in \{1, \dots, p-1\}$ and $\lambda_i > 0$ for $i \in \{p, \dots, n-1\}$.
Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}$ coincide at U and even such that $x'\big|_{\partial\Omega} = \overline{x}'\big|_{\partial\Omega}$.

Theorem 2.4.1. *Consider around U a local adapted coordinate system $x = (x', x^n)$ such that $dx^i = d\overline{x}^i$ at U (for i in $\{1, \dots, n-1\}$). There exists locally, in a neighborhood of $x = 0$, a $\mathcal{C}^\infty$ solution u_p^{wkb} to*

$$\begin{aligned} \Delta^{(p)}_{f,h} u_p^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (2.4.5)\\ \mathbf{t} u_p^{wkb} &= 0 \ \text{on } \partial\Omega && (2.4.6)\\ \mathbf{t} d^*_{f,h} u_p^{wkb} &= 0 \ \text{on } \partial\Omega \,, && (2.4.7) \end{aligned}$$

where u_p^{wkb} has the form:

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}} \,, \\ \text{with} \quad a(x,h) &\sim \sum_k a^k(x) h^k \ \text{and } a^0(0) = dx^1 \wedge \dots \wedge dx^{p-1} \wedge dx^n \,. \end{aligned}$$

2.5 First boundary conditions in the Dirichlet case

Writing

$$a(x,h) = a_I(x,h)\ dx^I = a_{I'}(x,h)dx^{I'} + a_{I_n}(x,h)dx^{I_n}\,,$$

the first boundary condition is equivalent to:

$$\forall k \in \mathbb{N},\ \forall I' \in \mathcal{I}'\,,\ a^k_{I'}(x',0) \equiv 0\,. \tag{2.5.1}$$

This paragraph specifies some consequences of these conditions.

2.5.1 About $\mathcal{L} + \mathcal{L}^*$

The next relation is obviously satisfied

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = -2\mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla(f+\Phi)} + \mathcal{L}^*_{\nabla(f+\Phi)}\,,$$

and using again Proposition A.2.3, write:

$$\mathcal{L}^*_{\nabla(f+\Phi)} + \mathcal{L}_{\nabla(f+\Phi)} = \mathcal{R}_4\,,$$

where $\mathcal{R}_4$ is a 0-th order differential operator.

Writing $\mathcal{R}_4 = \mathcal{R}_4^T + \mathcal{R}_4^N$, we deduce from Remark A.2.2 (since $a^k_I dx^I = a^k_{I_n} dx^{I_n}$ on the boundary),

$$\left\{\begin{array}{lcl} \mathbf{t}\left(\mathcal{R}_4(a^k_I dx^I)\right) & = & a^k_{I_n}(x',0)\mathcal{R}_4^N(dx^{I_n}) = \tilde{\ell}'_{I'}(x',0)dx^{I'} \\ \mathbf{n}\left(\mathcal{R}_4(a^k_I dx^I)\right) & = & a^k_{I_n}(x',0)\mathcal{R}_4^T(dx^{I_n})\,, \end{array}\right.$$

where the $\tilde{\ell}'_{I'}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$'s (for I' in $\mathcal{I}'$).

Moreover, $f+\Phi$ satisfies here the assumptions of Corollary A.2.5 (from (2.4.1)-(2.4.4)), then $\mathcal{R}_4^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_4^T(x',0) = \begin{pmatrix} & & 0 \\ & \mathcal{R}_4^{T'}(x') & \vdots \\ & & 0 \\ 0 & \cdots\quad 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa(x')\, Id\,,$$

where δ, κ are $\mathcal{C}^\infty$ functions which satisfy $\delta(0) = 0$,
$\kappa(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} + \varphi)(0)\right)$, and

$$\mathcal{R}_4^{T'}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega} + \varphi)(0)\right)\,.$$

2.5.2 Expression of the codifferential d^*

In order to make a study similar to the one done in Section 2.2 for the Neumann case, we need to work with d^* and then, to have a handy expression of this operator.

Set, for a differential form ω, in the coordinate system (x', x^n)

$$\nabla_i = \nabla_{x^i}\,,\ \mathbf{a}_i^*\omega = dx^i \wedge \omega\,,\ \text{and } \mathbf{a}_i\omega = \mathbf{i}_{\nabla x^i}\omega\,.$$

Then d and d^* write (see [CFKS] pp. 238-247):

$$d = \sum_{i=1}^{n} \mathbf{a}_i^* \nabla_i = -\sum_{i=1}^{n} (\nabla_i)^* \mathbf{a}_i^*\,, \tag{2.5.2}$$

$$d^* = -\sum_{i=1}^{n} \mathbf{a}_i \nabla_i\,. \tag{2.5.3}$$

Recall the characteristic relations:

$$\forall i,j \in \{1,\dots,n\}\,,\quad \mathbf{a}_i^*\mathbf{a}_j^* + \mathbf{a}_j^*\mathbf{a}_i^* = 0\,, \tag{2.5.4}$$

$$\mathbf{a}_i\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_i = 0\,, \tag{2.5.5}$$

$$\mathbf{a}_i^*\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_i^* = g^{ij}\,. \tag{2.5.6}$$

Denoting by ∂_i the operator defined by components with differentiation in a fixed coordinate system,

$$\partial_i(\omega_I dx^I) = \frac{\partial \omega_I}{\partial x^i} dx^I\,,$$

∇_i writes (see [CFKS] pp. 238-247)

$$\nabla_i = \partial_i - \sum_{j,l,m} \Gamma_{il}^j g_{jm} \mathbf{a}_l^* \mathbf{a}_m\,, \tag{2.5.7}$$

where the Γ_{il}^j are the Christoffel symbols.
Then d^* writes:

$$\begin{aligned} d^* &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \mathbf{a}_i \mathbf{a}_l^* \mathbf{a}_m \\ &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \left(\mathbf{a}_i\mathbf{a}_l^* + \mathbf{a}_l^*\mathbf{a}_i\right)\mathbf{a}_m - \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \mathbf{a}_l^*\mathbf{a}_i\mathbf{a}_m \\ &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} g^{il} \mathbf{a}_m - \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \mathbf{a}_l^*\mathbf{a}_i\mathbf{a}_m\,. \end{aligned} \tag{2.5.8}$$

2.5.3 Results

Proposition 2.5.1. *Using the notations of Appendix A.1 and Section 2.5.1, the next relations are satisfied for all k in $\mathbb{N}$, when* (2.5.1) *is fulfilled:*

$$\left\{ \begin{array}{lcl} \mathbf{t}\left((-2\mathcal{L}^*_{\nabla\Phi}+\mathcal{R}_4)a^k\right) & = & 2\left(\frac{\partial a^k_{I'}}{\partial x^n}\frac{\partial \Phi}{\partial x^n}+\ell_{I'}(x',0)\right)dx^{I'} \\ \mathbf{n}\left((-2\mathcal{L}^*_{\nabla\Phi}+\mathcal{R}_4)a^k\right) & = & (2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{Dir})a^k_{I_n}dx^{I_n}-2\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*a^k\,, \end{array}\right.$$

where the $\ell_{I'}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$'s (for I' in $\mathcal{I}'$) and $\mathcal{R}^T_{Dir}$ is a 0-th order differential operator given on the boundary by the next matrix, in the coordinates (x',x^n):

$$\mathcal{R}^T_{Dir}(x',0)=\begin{pmatrix} & & & 0 \\ & \mathcal{R}^{T'}_{Dir}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa_2(x')\,Id\,,$$

where $\delta(0)=0$,

$$\kappa_2(0)=Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,,$$

and

$$\mathcal{R}^{T'}_{Dir}(0)=2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$

Proposition 2.5.2. *Assume* (2.5.1) *for k in $\mathbb{N}$. The p-form*

$$\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*a^k$$

is then normal and the next equivalence is locally valid on the boundary $\partial\Omega$:

$$\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*a^k=0\Leftrightarrow \mathbf{t}d^*a^k=0\,.$$

Proof. Write indeed on the boundary $\partial\Omega$:

$$\begin{aligned} \frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*a^k &= \frac{\partial\Phi}{\partial x^n}dx^n\wedge\mathbf{n}d^*a^k+\frac{\partial\Phi}{\partial x^n}dx^n\wedge\mathbf{t}d^*a^k \\ &= 0+\frac{\partial\Phi}{\partial x^n}dx^n\wedge\mathbf{t}d^*a^k \\ &= \frac{\partial\Phi}{\partial x^n}dx^n\wedge(d^*a^k)_{I'}dx^{I'} \\ &= (-1)^{p-1}\frac{\partial\Phi}{\partial x^n}(d^*a^k)_{I'}dx^{I'}\wedge dx^n\,. \end{aligned}$$

Moreover, using

$$\frac{\partial f}{\partial n}(U) = -\frac{\partial \Phi}{\partial n}(U) > 0\,,$$

$\frac{\partial \Phi}{\partial x^n}$ is locally negative on the boundary which leads to the result.

∎

Lemma 2.5.3. *Under* (2.5.1)*, the next relations are satisfied for* k *in* $\mathbb{N}$*:*

$$\left\{\begin{array}{rcl} \mathbf{n}\left((\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}})a^k\right) & = & \frac{\partial\Phi}{\partial x^n} dx^n \wedge d^* a^k\,, \\ \mathbf{t}\left((\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}})a^k\right) & = & \left(-\frac{\partial a^k_{I'}}{\partial x^n}\frac{\partial \Phi}{\partial x^n} + \tilde{\ell}_{I'}(x',0)\right) dx^{I'}\,, \end{array}\right.$$

where the $\tilde{\ell}_{I'}$*'s are algebraically* $\mathcal{C}^\infty(\partial\Omega)$*-linear combinations of the* $a^k_{I_n}$*'s (for* I_n *in* $\mathcal{I}_n$*) which do not depend on the* $a^k_{I'}$*'s (for* I' *in* $\mathcal{I}_n$*).*

Proof. Owing to (1.2.5) and to the Cartan formula (1.2.8), write in the coordinates (x', x^n) (the function $\tilde{\Phi}$ is defined in Appendix A.1):

$$\begin{aligned} \left(\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}}\right) a^k &= d^*(d\Phi \wedge a^k) + d\Phi \wedge d^* a^k + d^*(d\tilde{\Phi} \wedge a^k) + d\tilde{\Phi} \wedge d^* a^k \\ &= d^*(\frac{\partial \Phi}{\partial x^n} dx^n \wedge a^k) + \frac{\partial \Phi}{\partial x^n} dx^n \wedge d^* a^k \\ &+ d^*((d_T\Phi - d\tilde{\Phi}) \wedge a^k) + (d_T\Phi - d\tilde{\Phi}) \wedge d^* a^k\,. \qquad (2.5.9) \end{aligned}$$

The second term $\frac{\partial\Phi}{\partial x^n} dx^n \wedge d^* a^k$ of the r.h.s. of (2.5.9) is normal according to Proposition 2.5.2.
Moreover, since $d_T\Phi = d\tilde{\Phi}$ on the boundary, the term $(d_T\Phi - d\tilde{\Phi}) \wedge d^* a^k$ of the r.h.s. also equals 0 on $\partial\Omega$.
Hence, write on $\partial\Omega$:

$$\begin{aligned} \left(\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}}\right) a^k &= \frac{\partial \Phi}{\partial x^n} dx^n \wedge d^* a^k \\ &+ d^*(\frac{\partial \Phi}{\partial x^n} dx^n \wedge a^k) + d^*((d_T\Phi - d\tilde{\Phi}) \wedge a^k)\,. \qquad (2.5.10) \end{aligned}$$

Study in a first time the term $d^*(\frac{\partial\Phi}{\partial x^n} dx^n \wedge a^k)$. Writing

$$a^k = a^k_I dx^I = a^k_{I'} dx^{I'} + a^k_{I_n} dx^{I_n}\,,$$

we deduce (in $\overline{\Omega}$):

$$\frac{\partial \Phi}{\partial x^n} dx^n \wedge a^k = \frac{\partial \Phi}{\partial x^n} a^k_{I'} dx^n \wedge dx^{I'}\,,$$

and, applying d^* to this last relation (see (2.5.8)), we obtain on $\partial\Omega$ (remember that $a^k_{I'} = 0$ on $\partial\Omega$)

$$\begin{aligned} d^*(\frac{\partial\Phi}{\partial x^n}dx^n \wedge a^k) &= -\sum_i \mathbf{a}_i\partial_i(\frac{\partial\Phi}{\partial x^n}a^k_{I'}dx^n \wedge dx^{I'}) \\ + \sum_{i,j,l,m}\Gamma^j_{il}g_{jm}g^{il}\mathbf{a}_m(\frac{\partial\Phi}{\partial x^n}a^k_{I'}\,dx^n \wedge dx^{I'}) &- \sum_{i,j,l,m}\Gamma^j_{il}g_{jm}\mathbf{a}^*_l\mathbf{a}_i\mathbf{a}_m(\frac{\partial\Phi}{\partial x^n}a^k_{I'}dx^n \wedge dx^{I'}) \\ &= -\sum_i \mathbf{a}_i\partial_i(\frac{\partial\Phi}{\partial x^n}a^k_{I'}dx^n \wedge dx^{I'}) + 0 \\ &= -\mathbf{i}_{\nabla x^n}\frac{\partial\Phi}{\partial x^n}\frac{\partial a^k_{I'}}{\partial x^n}dx^n \wedge dx^{I'} \\ &= -\frac{\partial\Phi}{\partial x^n}\frac{\partial a^k_{I'}}{\partial x^n}dx^{I'} \qquad (2.5.11) \end{aligned}$$

(we used at the last line the fact that G_0^{-1} is block diagonal with $g^{nn} \equiv 1$). Look now at the third term of the r.h.s. of (2.5.10) and write:

$$\begin{aligned} (d_T\Phi - d\tilde{\Phi}) \wedge a^k_I dx^I &= \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)a^k_I dx^i \wedge dx^I \\ &=: \sum_{i=1}^{n-1}\alpha_i a^k_I dx^i \wedge dx^I \end{aligned}$$

where, for i in $\{1,\dots,n-1\}$,

$$\alpha_i = \frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\,.$$

Hence, for j in $\{1,\dots,n-1\}$, α_j satisfies (see (A.1.5)):

$$\forall j \in \{1,\dots,n-1\}\ ,\alpha_j(x',0) \equiv 0\,.$$

We obtain consequently on $\partial\Omega$(see again (2.5.8)),

$$\begin{aligned} d^*((d_T\Phi - d\tilde{\Phi}) \wedge a_I^k dx^I)(x',0) &= -\sum_i \mathbf{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j a_I^k dx^j \wedge dx^I \\ + \left(\sum_{i,j,l,m} \Gamma_{il}^j g_{jm} g^{il} \mathbf{a}_m - \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \mathbf{a}_l^* \mathbf{a}_i \mathbf{a}_m \right) & \sum_{j=1}^{n-1} \alpha_j a_I^k dx^j \wedge dx^I \\ &= -\sum_i \mathbf{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j a_I^k dx^j \wedge dx^I \\ &= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n} (\alpha_j a_I^k) dx^j \wedge dx^I \end{aligned}$$

where we used $\alpha_j(x',0) \equiv 0$ at the two last lines.
Now, since $g^{ni} = g^{in} = 0$ for i in $\{1, \ldots, n-1\}$, write for all $I' \in \mathcal{I}'$:

$$\mathbf{a}_n dx^{I'} = \mathbf{i}_{\nabla x^n} dx^{I'} = 0 \,.$$

It implies:

$$\begin{aligned} d^*((d_T\Phi - d\tilde{\Phi}) \wedge a_I^k dx^I)(x',0) &= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n} (\alpha_j a_I^k) dx^j \wedge dx^I \\ &= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n} (\alpha_j a_{I_n}^k) dx^j \wedge dx^{I_n} \\ &= (-1)^{p+1} \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n} (\alpha_j a_{I_n}^k) dx^j \wedge dx^{I_n \setminus \{n\}} \\ &= (-1)^{p+1} \sum_{j=1}^{n-1} a_{I_n}^k \frac{\partial \alpha_j}{\partial x^n} (x',0) dx^j \wedge dx^{I_n \setminus \{n\}} \\ &=: \tilde{\ell}_{I'}(x',0) dx^{I'} \,, \end{aligned} \tag{2.5.12}$$

where the $\tilde{\ell}_{I'}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a_{I'}^k$'s (for I' in $\mathcal{I}'$).
Combining (2.5.10), (2.5.11), and (2.5.12) leads to the result announced in Lemma 2.5.3.

∎

Proof of Proposition 2.5.1.
Having in mind Lemma 2.5.3, look at the term $-2\mathcal{L}^*_{\nabla\tilde{\Phi}} + \mathcal{R}_4$.
Again by Proposition A.2.3, write:

$$\begin{aligned} -2\mathcal{L}^*_{\nabla\tilde{\Phi}} &= 2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_5 \\ &= 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_5 + \mathcal{R}_6 \end{aligned}$$

where $\mathcal{R}_5 = \mathcal{R}_5^T + \mathcal{R}_5^N$ and $\mathcal{R}_6 = \mathcal{R}_6^T + \mathcal{R}_6^N$ are 0-th order differential operators which satisfy, for $i \in \{5, 6\}$ (since $a_I^k dx^I = a_{I_n}^k dx^{I_n}$ on the boundary):

$$\left\{ \begin{array}{lcl} \mathbf{t}\left(\mathcal{R}_i(a_I^k dx^I)\right) &=& a_{I_n}^k(x', 0)\mathcal{R}_i^N(dx^{I_n}) = \tilde{\ell}_{I'}^{i'}(x', 0)dx^{I'} \\ \mathbf{n}\left(\mathcal{R}_i(a_I^k dx^I)\right) &=& a_{I_n}^k(x', 0)\mathcal{R}_i^T(dx^{I_n})\,, \end{array} \right.$$

where the $\tilde{\ell}_{I'}^{i'}(x', 0)$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a_{I'}^k$'s (for I' in $\mathcal{I}'$).
Moreover, since $\tilde{\Phi}$ satisfies the assumptions of Corollary A.2.5, $\mathcal{R}_5^T$ and $\mathcal{R}_6^T$ are given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_5^T = \left(\begin{array}{ccc|c} & & & 0 \\ & \mathcal{R}_5^{T'} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{array} \right)^{(p)} - \zeta(x')\, Id \ \text{ and } \mathcal{R}_6^T = \left(\begin{array}{ccc|c} & & & 0 \\ & \mathcal{R}_6^{T'} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{array} \right)^{(p)},$$

where

$$\zeta(0) = -2\,Tr\left(\mathrm{Hess}\,(\tilde{\Phi}\big|_{\partial\Omega})(0)\right) = -2\,Tr\left(\mathrm{Hess}\,(\varphi)(0)\right)\,,$$

and $\mathcal{R}_5^{T'}$, $\mathcal{R}_6^{T'}$ write at 0:

$$\mathcal{R}_5^{T'}(0) = -4\left(\mathrm{Hess}\,(\varphi)(0)\right) \ \text{ and } \mathcal{R}_6^{T'}(0) = 2\left(\mathrm{Hess}\,(\varphi)(0)\right)$$

Set $\mathcal{R}_{\mathrm{Dir}} = \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6$ and $\tilde{\ell}_{I'}^{(3)} = \tilde{\ell}'_{I'} + \tilde{\ell}_{I'}^{5'} + \tilde{\ell}_{I'}^{6'}$ for I' in $\mathcal{I}'$. $\mathcal{R}_{\mathrm{Dir}}$ is a 0-th order differential operator which satisfies

$$-2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_4 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_{\mathrm{Dir}} \tag{2.5.13}$$

and

$$\left\{ \begin{array}{lcl} \mathbf{t}\left(\mathcal{R}_{\mathrm{Dir}}(a_I^k dx^I)\right) &=& a_{I_n}^k(x', 0)\mathcal{R}_{\mathrm{Dir}}^N(dx^{I_n}) = \tilde{\ell}_{I'}^{(3)}(x', 0)dx^{I'} \\ \mathbf{n}\left(\mathcal{R}_{\mathrm{Dir}}(a_I^k dx^I)\right) &=& a_{I_n}^k(x', 0)\mathcal{R}_{\mathrm{Dir}}^T(dx^{I_n})\,, \end{array} \right. \tag{2.5.14}$$

where the $\tilde{\ell}_{I'}^{(3)}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I_n}^k$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a_{I'}^k$'s (for I' in $\mathcal{I}'$).
Moreover, $\mathcal{R}_{\mathrm{Dir}}^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}^{T}_{\mathrm{Dir}}(x',0)=\begin{pmatrix} & & & 0\\ & \mathcal{R}^{T'}_{\mathrm{Dir}}(x',0) & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & \delta(x')\end{pmatrix}^{(p)}-\kappa_2(x')Id\,,$$

where $\delta(0)=0$,

$$\begin{aligned}\kappa_2(0)=\kappa(0)+\zeta(0) &= Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}+\varphi)(0)\right)-2\,Tr\,(\mathrm{Hess}\,(\varphi)(0))\\ &= Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^{T'}_{\mathrm{Dir}}(0) &= \mathcal{R}^{T'}_{4}(0)+\mathcal{R}^{T'}_{5}(0)+\mathcal{R}^{T'}_{6}(0)\\ &= 2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}+\varphi)(0)\right)-2\,(\mathrm{Hess}\,(\varphi)(0))\\ &= 2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.\end{aligned}$$

Look now at the term $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id$. By the Cartan formula (1.2.8),

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)a^k \;=\; da^k_{I'}(\nabla\tilde{\Phi})dx^{I'}+da^k_{I_n}(\nabla\tilde{\Phi})dx^{I_n}\,,$$

and, using the boundary conditions satisfied by the a^k_I's (for I in $\mathcal{I}$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain:

$$\begin{aligned}(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)a^k &= \sum_{i=1}^{n-1}\frac{\partial a^k_{I_n}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I_n}\\ &= 2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id\,a^k_{I_n}dx^{I_n}\,.\end{aligned}\tag{2.5.15}$$

Set $\ell_{I'}=-\tilde{\ell}_{I'}+\frac{1}{2}\tilde{\ell}^{(3)}_{I'}$ and write

$$\left(-2\mathcal{L}^*_{\nabla\Phi}+\mathcal{R}_3\right)a^k=-2\left(\mathcal{L}^*_{\nabla\Phi}-\mathcal{L}^*_{\nabla\tilde{\Phi}}\right)a^k+\left(-2\mathcal{L}^*_{\nabla\tilde{\Phi}}+\mathcal{R}_3\right)a^k\,.$$

Using (2.5.13), (2.5.14), (2.5.15), Proposition 2.5.1 is then a direct consequence of Lemma 2.5.3.

∎

2.6 Proof of Theorem 2.4.1

Although the calculations are different, the scheme of the proof is the same as for Theorem 2.1.1. Consider first a WKB-approximation for

$$(\Delta_{f,h}^{(p)} - E(h))u_p^{wkb} = e^{-\frac{\Phi}{h}}\mathcal{O}(h^\infty) \tag{2.6.1}$$

with $E(h) = O(h^2)$ and the boundary conditions (2.4.6)(2.4.7).

From

$$\forall k \in \mathbb{N}\,, \quad d_{f,h}^*(e^{-\frac{\Phi}{h}}a^k) = e^{-\frac{\Phi}{h}}\left[hd^*a^k + \mathbf{i}_{\nabla(f+\Phi)}a^k\right]\,,$$

where a^k is a normal form and $\nabla(f+\Phi)$ is a tangential vectorfield (due to (2.4.6) and (2.4.3)), the second boundary condition corresponds to

$$\mathbf{t}(d^*a^k) = 0\,. \tag{2.6.2}$$

Recall now, using the notations of Appendix A.2.2 and Section 2.5.1, the next relation,

$$\begin{aligned} e^{\frac{\Phi}{h}}\Delta_{f,h}e^{-\frac{\Phi}{h}} &= h^2(d+d^*)^2 + h(2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}) \\ &= h^2(d+d^*)^2 + h(-2\mathcal{L}_{\nabla\Phi}^* + \mathcal{R}_4)\,. \end{aligned}$$

By looking for $E(h) \sim \sum_{k=1}^{\infty} h^{k+1}E_k$, the interior equation (2.6.1) reads, like in Section 2.3,

$$e^{\frac{\Phi}{h}}(\Delta_{f,h} - E(h))e^{-\frac{\Phi}{h}} = h^2[(d+d^*)^2 - h^{-2}E(h)] + h\left[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}\right]\,.$$

Hence, like in Section 2.3, the construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k's are known on $\partial\Omega$ (since the non characteristic Cauchy problems

$$[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}]a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega}\,. \tag{2.6.3}$$

are solved by induction with the convention $a_{-1} = 0$)
and the problem is reduced to the solving of the system made of the boundary conditions (2.5.1), (2.6.2) and of the compatibility equation (see Section 2.5.1 for the meaning of the notations):

$$[-2\mathcal{L}_{\nabla\Phi}^* + \mathcal{R}_4]\,a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{on } \partial\Omega\,. \tag{2.6.4}$$

Owing to Propositions 2.5.1 and 2.5.2 (with the notations of Section 2.5.3) and to (2.4.3), the system (2.6.4), (2.5.1), (2.6.2) is equivalent to the differential system on $\partial\Omega$:

$$
\begin{cases}
-\mathbf{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{\mathrm{Dir}}) a^k_{I_n} dx^{I_n} & (2.6.5) \\
-\mathbf{t}(d+d^*)^2 a^{k-1} - 2\ell_{I'}(x',0) dx^{I'} = -2 \frac{\partial f}{\partial n} \frac{\partial a^k_{I'}}{\partial x^n} dx^{I'} & (2.6.6) \\
(2.5.1) \ + \ (2.6.2)\,,
\end{cases}
$$

where the $\ell_{I'}$'s are algebraically $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$'s (for I' in $\mathcal{I}'$).
Moreover, since $dx^i = d\overline{x}^i$ (for $i \in \{1, \ldots, n-1\}$) at the point U, thanks to Corollary A.2.5, (2.1.5)-(2.1.6), and according to [HeSj4] pp. 271-274, $\mathcal{R}^T_{\mathrm{Dir}}(0)$ restricted to normal forms is symmetric with the one dimensional kernel $\mathbb{R}dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n$.

Since $a^k_{I_n} dx^{I_n}$ is normal and $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id$ only differentiates tangentially the $a^k_{I_n}$'s

$$
\left(\quad \text{since } (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id) a^k_{I_n} dx^{I_n} = \sum_{i=1}^{n-1} \frac{\partial a^k_{I_n}}{\partial x^i} (\nabla\tilde{\Phi})_i dx^{I_n} \ \right),
$$

(2.6.5) can be rewritten as a tangential system which can be solved according to the analysis of the boundaryless case done in [HeSj4].
Here are the details:
Owing to Proposition 2.5.2, the complete system becomes equivalent to

$$
\begin{cases}
(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{\mathrm{Dir}}) a^k_{I_n} dx^{I_n} = -\mathbf{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell} + E_k a^0 & \text{on } \partial\Omega \\
(2\mathcal{L}_{\nabla\Phi} \otimes Id + \mathcal{R}) a^k = -(d+d^*)^2 a^{k-1} & \text{on } \overline{\Omega} \\
\qquad \forall I' \in \mathcal{I}'\,, \ a_{I'}|_{\partial\Omega} \equiv 0\,. &
\end{cases}
$$

The first line is again a degenerate matricial transport equation which can be solved according to [HeSj4][Hel2]:
For $k=0$, take $a^0(0) = dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n \in \ \mathrm{Ker}\, (\mathcal{R}^T_{\mathrm{Dir}}(0))$ and for $k > 0$ choose E_k so that the compatibility condition

$$
-\mathbf{n}(d+d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0) \in \left(\mathrm{Ker}\, (\mathcal{R}^T_{\mathrm{Dir}}(0)) \right)^\perp
$$

is satisfied. Thus, at every step $k \in \mathbb{N}$, the first and the third line of the previous system fully determine the Cauchy data $a^k(x',0)$ and the number

E_k. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Proposition 2.5.2, the second trace condition (2.6.2).
Checking $E(h) = O(h^\infty)$ is then identical to the end of the proof of Theorem 2.1.1 done in Section 2.3 after choosing a cut-off function χ which satisfies $\nabla\chi = \nabla_T\chi$ on the boundary $\partial\Omega$. ∎

A Computations in local adapted coordinate systems

We work here in a local adapted coordinate system (x', x^n) around $U \in \partial\Omega$ in order to apply indifferently the results of this section to the Neumann and Dirichlet cases.

A.1 A modified Agmon distance

Define $\tilde{\Phi}$ around U in the coordinates (x', x^n) by

$$\forall x = (x', x^n)\,,\ \tilde{\Phi}(x', x^n) = \Phi(x', 0)\,, \tag{A.1.1}$$

and note the next relation satisfied for all x around U, in the coordinates (x', x^n), due to the form of $G_0^{\pm 1}$ (see Remark 1.3.3):

$$\left\{\begin{array}{lcl} d\tilde{\Phi}(x) & = & d_T\tilde{\Phi}(x) + \frac{\partial\tilde{\Phi}}{\partial x^n}(x)dx^n = d_T\tilde{\Phi}(x) \\ \nabla\tilde{\Phi}(x) & = & \nabla_T\tilde{\Phi}(x) + \frac{\partial\tilde{\Phi}}{\partial x^n}(x)\frac{\partial}{\partial x^n} = \nabla_T\tilde{\Phi}(x) \end{array}\right. .$$

For a vector (or a vector field) $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x^i}$, making the identification

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

the tangential part X_T (resp. the normal part X_N) of X is defined as:

$$X_T = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \\ 0 \end{pmatrix} \text{ (resp. } X_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_n \end{pmatrix}).$$

Similarly, for a (n,n)-matrix $A(x) = (a_{ij}(x))_{i,j}$, define $A_T(x)$ and $A_N(x)$ by:

$$A_T = \begin{pmatrix} & & & 0 \\ & A' & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \text{ and } A_N = \begin{pmatrix} & & & a_{1n} \\ & [0] & & \vdots \\ & & & a_{n-1\,n} \\ a_{n1} & \cdots & a_{n\,n-1} & 0 \end{pmatrix}.$$

Recall moreover that, for a vector (or a vector field) X and a $\mathcal{C}^\infty$ function ψ, the identification $\langle \nabla\psi \,|\, X\rangle_{g_0} = d\psi(X)$ leads to:

$$\nabla\psi = G_0^{-1} \begin{pmatrix} \frac{\partial\psi}{\partial x^1} \\ \vdots \\ \frac{\partial\psi}{\partial x^n} \end{pmatrix}.$$

Hence, due to the form of G_0^{-1} (see Remark 1.3.3), the next relations are indeed satisfied:

$$\begin{aligned} (\nabla\psi)_T &= \nabla_T\psi = G_0^{-1} \begin{pmatrix} \frac{\partial\psi}{\partial x^1} \\ \vdots \\ \frac{\partial\psi}{\partial x^{n-1}} \\ 0 \end{pmatrix} \\ &\text{and} \\ (\nabla\psi)_N &= \frac{\partial\psi}{\partial x^n}\frac{\partial}{\partial x^n} = G_0^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial\psi}{\partial x^n} \end{pmatrix}. \end{aligned}$$

In the Neumann case, we are going to compare $\mathcal{L}_{\nabla\Phi}$ and $\mathcal{L}_{\nabla\tilde{\Phi}}$ and the next relations could be convenient:

$$\nabla\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ \frac{\partial\Phi}{\partial x^n}(x) \end{pmatrix} \tag{A.1.2}$$

and

$$\nabla_T\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ 0 \end{pmatrix}. \tag{A.1.3}$$

At least, we are going to compare $\mathcal{L}^*_{\nabla\Phi}$ and $\mathcal{L}^*_{\nabla\tilde{\Phi}}$ in the Dirichlet case and the next relations could also be convenient:

$$d\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)dx^i + \frac{\partial\Phi}{\partial x^n}(x)dx^n \quad \text{(A.1.4)}$$

and

$$d_T\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)dx^i\,. \quad \text{(A.1.5)}$$

A.2 About $\mathcal{L} + \mathcal{L}^*$

A.2.1 For a general $\mathcal{C}^\infty$ function h

In this subsection, we give similar results to those done in [HeSj4] Appendix A.

Take h a $\mathcal{C}^\infty$ function from $\overline{\Omega}$ on $\mathbb{R}$ and write:

$$\nabla h = \sum_{i=1}^{n}(\nabla h)_i \frac{\partial}{\partial x^i}\,.$$

According to [HeSj4], give the next algebraic definition:

Definition A.2.1. *For a Euclidean space $(E, \langle\cdot\,|\,\cdot\rangle)$ and $A \in \mathcal{L}(E)$, $A^{(p)}$ and $\Gamma^{(p)}(A)$ denote respectively the linear application $A^{(p)} \in \mathcal{L}(\Lambda^p E)$ and the application $\Gamma^{(p)}(A) = A \otimes \cdots \otimes A$:*

$$A^{(p)}(\omega_1 \wedge \cdots \wedge \omega_p) = (A\omega_1 \wedge \cdots \wedge \omega_p) + \cdots + (\omega_1 \wedge \cdots \wedge A\omega_p)$$

and

$$\Gamma^{(p)}(A)(\omega_1 \wedge \cdots \wedge \omega_p) = (A\omega_1) \wedge \cdots \wedge (A\omega_p)\,.$$

(with the obvious convention $A^{(0)} = 0$ and $\Gamma^{(0)}(A) = 1$).

Remark A.2.2. *Under the canonical identification $\Lambda^1 E = E$, note that $A^{(1)} = A$. Moreover, if A^* denotes the adjoint of A according to the scalar product on E, the adjoint of $A^{(p)}$ is simply $(A^{(p)})^* = (A^*)^{(p)} =: A^{(p),*}$ (recall that $\Lambda^p E$ is a Euclidean space with the scalar product $\langle\cdot\,|\,\cdot\rangle_p$:*

$$\langle \omega_1 \wedge \cdots \wedge \omega_p \,|\, \mu_1 \wedge \cdots \wedge \mu_p\rangle_p = \det\left(\left(\langle\omega_i \,|\, \mu_j\rangle\right)_{i,j}\qquad\right).$$

Remark that for a p-form $a_I^k dx^I = a_{I'}^k dx^{I'} + a_{I_n}^k dx^{I_n}$, with the notations of Appendix A.1, $A^{(p)} = A_T^{(p)} + A_N^{(p)}$ and:

$$\left\{\begin{array}{lcl} \mathbf{t}\left(A^{(p)}(a_I^k dx^I)\right) & = & a_{I'}^k(x',0)A_T^{(p)}(dx^{I'}) + a_{I_n}^k(x',0)A_N^{(p)}(dx^{I_n}) \\ \mathbf{n}\left(A^{(p)}(a_I^k dx^I)\right) & = & a_{I_n}^k(x',0)A_T^{(p)}(dx^{I_n}) + a_{I'}^k(x',0)A_N^{(p)}(dx^{I'})\,. \end{array}\right.$$

Moreover, for any 0-th order differential operator $\mathcal{A}$ on the form $\mathcal{A} = A^{(p)} + \psi\, Id$, where ψ is a $\mathcal{C}^\infty$ function, we will denote by $\mathcal{A}^T$ and $\mathcal{A}^N$ the next 0-th order differential operators:

$$\mathcal{A}^T = A_T^{(p)} + \psi\, Id \quad \text{and} \quad \mathcal{A}^N = A_N^{(p)}$$

(notice that $\mathcal{A}^T$ (resp. $\mathcal{A}^N$) coincides with $A_T^{(p)}$ (resp. $A_N^{(p)}$) if $\psi \equiv 0$).
Furthermore, our aim is to work with tangential forms in the Neumann case (i.e. $a_I^k dx^I = a_{I'}^k dx^{I'}$ on $\partial\Omega$) and with normal forms in the Dirichlet case (i.e. $a_I^k dx^I = a_{I_n}^k dx^{I_n}$ on $\partial\Omega$). Hence, for any tangential form in the Neumann case (resp. for any normal form in the Dirichlet case), write:

$$\left\{\begin{array}{lcl} \mathbf{t}\left(\mathcal{A}(a_I^k dx^I)\right) & = & a_{I'}^k(x',0)A_T^{(p)}(dx^{I'}) + \psi(x',0)a_{I'}^k(x',0)dx^{I'} \\ & = & \mathbf{t}\left(\mathcal{A}^T(a_I^k dx^I)\right) \\ & & \\ \mathbf{n}\left(\mathcal{A}(a_I^k dx^I)\right) & = & a_{I'}^k(x',0)A_N^{(p)}(dx^{I'}) = \mathbf{n}\left(\mathcal{A}^N(a_I^k dx^I)\right) \end{array}\right. \tag{A.2.1}$$

(resp.

$$\left\{\begin{array}{lcl} \mathbf{t}\left(\mathcal{A}(a_I^k dx^I)\right) & = & a_{I_n}^k(x',0)A_N^{(p)}(dx^{I'}) = \mathbf{t}\left(\mathcal{A}^N(a_I^k dx^I)\right) \\ & & \\ \mathbf{n}\left(\mathcal{A}(a_I^k dx^I)\right) & = & a_{I_n}^k(x',0)A_T^{(p)}(dx^{I_n}) + \psi(x',0)a_{I_n}^k(x',0)dx^{I_n} \\ & = & \mathbf{n}\left(\mathcal{A}^T(a_I^k dx^I)\right) \end{array}\right.).\tag{A.2.2}$$

The end of this section is devoted to the proof of the next proposition:

Proposition A.2.3. *In the coordinates (x', x^n), the next equalities are satisfied:*

$$\left\{\begin{array}{ll} \mathcal{L}_{\nabla h} & = \mathcal{L}_{\nabla h} \otimes Id + \mathcal{R}_h \\ \mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^* & = \mathcal{R}_h + \mathcal{R}_h^* - \left(\sum_{i=1}^n \left(\frac{\partial(\nabla h)_i}{\partial x^i} + \frac{1}{2}(\nabla h)_i \frac{\partial[\det G_0]}{\partial x^i}\right)\right) Id \\ & \qquad - \sum_{i=1}^n (\nabla h)_i (G_0 \frac{\partial[G_0^{-1}]}{\partial x^i})^{(p)}\,, \end{array}\right.$$

where $(\mathcal{L}_{\nabla h} \otimes Id)a_I^k dx^I = (\mathcal{L}_{\nabla h}(a_I^k))dx^I$, $\mathcal{R}_h$ *is the* 0*-th differential operator given by the matrix:*

$$\mathcal{R}_h(x) = \left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)_{i,j}^{(p)} =: A_h^{(p)}\,,$$

and $\left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)_{i,j}$, $G_0\frac{\partial[G_0^{-1}]}{\partial x^i}$ *are viewed as endomorphisms of* $T_x^*\overline{\Omega}$. *Moreover,* $\mathcal{R}_h^*$ *is given by the matrix:*

$$\mathcal{R}_h^* := A_h^{(p),*} = (G_0{}^t\!A_h G_0^{-1})^{(p)}\,.$$

Remark A.2.4. *From the computations done in Appendix A.1, write* $(\nabla h)_n = \frac{\partial h}{\partial x^n}$. *Moreover, due to the form of* $G_0^{\pm 1}$, *note that*

$$\mathcal{R}_h + \mathcal{R}_h^* - \sum_{i=1}^{n}(\nabla h)_i(G_0\frac{\partial[G_0^{-1}]}{\partial x^i})^{(p)}$$

is given by the matrix:

$$\begin{pmatrix} A_h' + G_0'{}^t\!A_h' G_0^{-1\prime} - \sum_{i=1}^n(\nabla h)_i G_0'\frac{\partial[G_0^{-1\prime}]}{\partial x^i} & \left(\frac{\partial^2 h}{\partial x^n\partial x^i}\right)_{i,1} + G_0'\left(\frac{\partial(\nabla h)_i}{\partial x^n}\right)_{i,1} \\ \left(\frac{\partial(\nabla h)_j}{\partial x^n}\right)_{1,j} + \left(\frac{\partial^2 h}{\partial x^n\partial x^j}\right)_{1,j} G_0^{-1\prime} & \frac{\partial^2 h}{\partial (x^n)^2} \end{pmatrix}^{(p)}.$$

Corollary A.2.5. *In the coordinates* (x', x^n), *assume that the function* h *admits a critical point at* 0, *that* $\frac{\partial h}{\partial x^n} \equiv 0$ *on the boundary* $\partial\Omega$, *and that* $\frac{\partial^2 h}{\partial(x^n)^2}(0) = 0$. *Then the next relations are true:*

$$\mathcal{R}_h(0) = \mathcal{R}_h^*(0) = \begin{pmatrix} & & 0 \\ & \mathrm{Hess}\,(h\big|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)}$$

and

$$\left(\mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^*\right)(0) = 2\mathcal{R}_h(0) - Tr\left(\mathrm{Hess}\,(h\big|_{\partial\Omega})(0)\right) Id\,.$$

Proof. Since (x', x^n) are local adapted coordinates around $U \cong 0$ and 0 is a critical point of h, note first that for all i in $\{1, \ldots, n\}$,

$$(\nabla h)_i = \sum_{j=1}^n g^{ij}\frac{\partial h}{\partial x^j} = \frac{\partial h}{\partial x^i} + \mathcal{O}(|x|^2)\,.$$

This implies

$$\mathcal{R}_h(x) = \left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)_{i,j}^{(p)} = (\mathrm{Hess}\,(h))^{(p)} + \mathcal{O}(|x|)$$

and in particular at 0, since $\frac{\partial h}{\partial x^n} \equiv 0$ on the boundary and $\frac{\partial^2 h}{\partial (x^n)^2}(0) = 0$:

$$\mathcal{R}_h(0) = \begin{pmatrix} & & 0 \\ & \mathrm{Hess}\,(h\big|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)} .$$

Moreover, we deduce from $G_0^{\pm 1}(0) = I_n$ and the symmetry of $\mathrm{Hess}\,(h\big|_{\partial\Omega})(0)$,

$$\mathcal{R}_h^*(0) = \mathcal{R}_h(0) .$$

At least, we obtain from $\frac{\partial^2 h}{\partial (x^n)^2}(0) = 0$,

$$-\left(\sum_{i=1}^n \frac{\partial (\nabla h)_i}{\partial x^i}\right) Id = -\,Tr\left(\mathrm{Hess}\,(h\big|_{\partial\Omega})(0)\right) \text{ at } 0 ,$$

which leads to the end of the proof, using that for all i in $\{1, \ldots, n\}$, $(\nabla h)_i(0) = \frac{\partial h}{\partial x^i}(0) = 0$. ∎

Proof of Proposition A.2.3.
The first equality is proved in [HeSj4] pp. 334-336. There is also a proof of the second equality in [HeSj4] but we need to be more precise here.
From the first equality, deduce:

$$\mathcal{L}_{\nabla h}^* = (\mathcal{L}_{\nabla h} \otimes Id)^* + \mathcal{R}_h^* .$$

Remarking that the scalar product of two p-forms ω and η is given by

$$\langle \omega \,|\, \eta \rangle_{g_0} = \langle \omega \,|\, \Gamma^{(p)}(G_0^{-1})\eta \rangle_{g_e} ,$$

where g_e is the Euclidean metric $\sum_{i=1}^n d(x^i)^2$, we obtain

$$\mathcal{R}_h^* = \Gamma^{(p)}(G_0)({}^t\!A_h)^{(p)}\Gamma^{(p)}(G_0^{-1}) = (G_0 {}^t\!A_h G_0^{-1})^{(p)} .$$

Look now at the term $(\mathcal{L}_{\nabla h} \otimes Id)^*$.
Take first two p-forms $\alpha\omega$ and $\beta\eta$ where α, β are $\mathcal{C}_0^\infty(\Omega, \mathbb{R})$ functions, and ω, η are two p-forms dx^I and dx^J. Denoting by $V_{g_0}(dx)$ the normalized volume form, $V_{g_0}(dx)$ satisfies:

$$V_{g_0}(dx) = (\det G_0(x))^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n =: \nu(x) dx^1 \wedge \cdots \wedge dx^n .$$

Hence we deduce:

$$\begin{aligned}\langle \alpha\omega \,|\, (\mathcal{L}_{\nabla h} \otimes Id)^* \beta\eta\rangle_{g_0} &= \langle \mathcal{L}_{\nabla h}(\alpha)\omega \,|\, \eta\rangle_{g_0} \\ &= \int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \,|\, \eta\rangle_{g_0(x)} (\det G_0(x))^{\frac{1}{2}} dx\,.\end{aligned}$$

Using now the Cartan formula (1.2.8), $\mathcal{L}_{\nabla h}(\alpha) = d\alpha(\nabla h) = \sum_{i=1}^n \frac{\partial \alpha}{\partial x^i}(\nabla h)_i$ and we obtain:

$$\begin{aligned}\int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \,|\, \eta\rangle_{g_0(x)} (\det G_0(x))^{\frac{1}{2}} dx &= \int \left(\sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i \beta \right) \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \, dx \\ &= -\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \big) dx\end{aligned}$$

Moreover, write:

$$\begin{aligned}\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \big) dx &= -\int \alpha \sum_{i=1}^n \left(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \right) dx \\ -\int \alpha \sum_{i=1}^n \left((\nabla h)_i \frac{\partial \beta}{\partial x^i} \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \right) dx &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \,|\, \eta\rangle_{g_0(x)}) \nu \right) dx \\ &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \right) dx \\ &= -\int \alpha \sum_{i=1}^n \left(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu \right) dx \\ -\int \alpha (\mathcal{L}_{\nabla h}(\beta)) \langle \omega \,|\, \eta\rangle_{g_0(x)} \nu dx &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \,|\, \eta\rangle_{g_0(x)}) \nu \right) dx \\ &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \langle \omega \,|\, \eta\rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \right) dx\,.\end{aligned}$$

Noting that for all i in $\{1, \ldots, n\}$,

$$\begin{aligned}\frac{\partial}{\partial x^i} \Gamma^{(p)}(G_0^{-1}) &= \Big(\frac{\partial G_0^{-1}}{\partial x^i} \otimes G_0^{-1} \otimes \cdots \otimes G_0^{-1}\Big) + \cdots + \Big(G_0^{-1} \otimes \cdots \otimes G_0^{-1} \otimes \frac{\partial G_0^{-1}}{\partial x^i}\Big) \\ &= \Gamma^{(p)}(G_0^{-1}) \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i}\Big)^{(p)},\end{aligned}$$

we deduce for all i in $\{1, \ldots, n\}$,

$$\frac{\partial}{\partial x^i} \langle \omega \,|\, \eta\rangle_{g_0(x)} = \langle \omega \,|\, \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i}\Big)^{(p)} \eta\rangle_{g_0(x)}\,.$$

Consequently,

$$\begin{aligned}(\mathcal{L}_{\nabla h}\otimes Id)^* &= -\mathcal{L}_{\nabla h}\otimes Id - \left(\sum_{i=1}^{n}\left(\frac{\partial(\nabla h)_i}{\partial x^i}+\frac{(\nabla h)_i}{\nu}\frac{\partial \nu}{\partial x^i}\right)\right) Id \\ &\quad -\sum_{i=1}^{n}(\nabla h)_i (G_0\frac{\partial[G_0^{-1}]}{\partial x^i})^{(p)}\,,\end{aligned}$$

which leads to the second result of Proposition A.2.3.

∎

A.2.2 Application to $\mathcal{L}_{\nabla\Phi}-\mathcal{L}^*_{\nabla\Phi}+\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}$

Write first

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}^*_{\nabla\Phi}+\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}=2\mathcal{L}_{\nabla\Phi}+\mathcal{L}_{\nabla(f-\Phi)}+\mathcal{L}^*_{\nabla(f-\Phi)}\,.$$

By Proposition A.2.3, we deduce the next relation:

$$\mathcal{L}^*_{\nabla(f-\Phi)}+\mathcal{L}_{\nabla(f-\Phi)}=\mathcal{R}_1\,,$$

where $\mathcal{R}_1$ is a 0-th order differential operator.

Furthermore, using now the first equality of Proposition A.2.3,

$$2\mathcal{L}_{\nabla\Phi}=2\mathcal{L}_{\nabla\Phi}\otimes Id+\mathcal{R}_2\,,$$

where $\mathcal{R}_2$ is a 0-th differential operator too.

Consequently, setting $\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2$, we obtain the following relation:

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}^*_{\nabla\Phi}+\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}=2\mathcal{L}_{\nabla\Phi}\otimes Id+\mathcal{R}\,,$$

where $\mathcal{R}$ is a 0-th order differential operator.

Acknowledgement: The author would like to thank F. Nier and T. Jecko for profitable discussions.

References

[ChLi] K. C. Chang and J. Liu. A cohomology complex for manifolds with boundary. Topological Methods in Non Linear Analysis. Volume 5, p. 325-340 (1995).

[CFKS] H.L Cycon, R.G Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Text and Monographs in Physics. Springer Verlag (1987).

[DiSj] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the semi-classical limit*. London Mathematical Society. Lecture Note Series 268. Cambridge University Press (1999).

[Duf] G.F.D. Duff. Differential forms in manifolds with boundary. Ann. of Math. 56, p. 115-127 (1952).

[DuSp] G.F.D. Duff and D.C. Spencer. Harmonic tensors on Riemannian manifolds with boundary. Ann. of Math. 56, p. 128-156 (1952).

[GHL] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry.* Universitext, 2nd Edition, Springer Verlag (1993).

[Gol] S.I. Goldberg. *Curvature and Homology.* Dover books in Mathematics, 3rd edition (1998).

[Hel2] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications.* Springer Verlag. Lecture Notes in Mathematics 1336 (1988).

[HKN] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea, 26, pp. 41-85 (2004).

[HeNi1] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary, Matematica Contemporanea Vol.**26** pp. 41-85, 2004.

[HeSj1] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit I, Comm. Partial Differential Equations 9 (4), p. 337-408, (1984).

[HeSj4] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique IV -Etude du complexe de Witten -. Comm. Partial Differential Equations 10 (3), p. 245-340 (1985).

[Lep] D. Le Peutrec. Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian. IRMAR. Preprint (2008).

[Mil1] J.W. Milnor. *Morse Theory.* Princeton University press (1963).

[Pet] P. Petersen. *Riemannian geometry.* Graduate Texts in Mathematics, Springer Verlag (1998).

[Sch] G. Schwarz. *Hodge decomposition. A method for Solving Boundary Value Problems.* Lecture Notes in Mathematics 1607, Springer Verlag (1995).

[Wit] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, p. 661-692 (1982).

[Zha] W. Zhang. *Lectures on Chern-Weil theory and Witten deformations.* Nankai Tracts in Mathematics. Vol. 4. World Scientific (2002).